CHAPTER I I - BAICA'S GENERAL EUCLIDEAN ALGORITHM (BGEA)

Section 0. Introduction.

In 1980, Baica [1] defined a modification of the (JPA) that used the Hasse and Bernstein initial vector, but was not restricted to the real numbers. For the first time the complex numbers were considered. The only differences in the definitions stated alone are that the Di's are now complex numbers. An immediate consequence of these extensions is that the bounds on D in the (HBA) are now eliminated and only the divisibility condition, $d \mid D$, remains. Returning to the example cited in the section 4 of chapter I it can now be seen that $w = \sqrt[5]{12^5 + 6}$ has a periodic development, only $6 \mid 12$ is required. At that time Baica proved only that $d \mid D$ is a necessary condition to make her algorithm to be periodic and named her algorithm, the Algorithm for Complex Fields (ACF). Later when Baica proved that $d \mid D$ is also a sufficient condition for the periodicity of her (ACF) algorithm then (ACF) became Baica's General Euclidean Algorithm (BGEA).

Section 1. The proof of the necessary condition for the periodicity of (BGEA).

1^*. The (BGEA) definition, basic properties. We denote :

(1^*.1) $S_{n+1} = \{d, D_1, D_2, \ldots, D_n\}, \quad n \geq 2,$

a set of any $n + 1$ algebraic numbers ;

(1^*.2) $\boldsymbol{K}_1 = Q(d, D_1, D_2, \ldots, D_n)$

the field generated by adjoining S_{n+1} to Q ;

(1^*.3) $$\begin{cases} G(x) = \sum_{i=0}^{n} c_i x^{n-i};\ c_0 = 1; c_i \in \hat{\boldsymbol{E}}_1,\ i = 1,\ldots,n \\ \text{an irreductible polynomialin } x \text{ over } \hat{\boldsymbol{E}}_1 \end{cases}$$

(1^*.4) $$\begin{cases} \hat{\boldsymbol{E}}_2 = \hat{\boldsymbol{E}}_1(w);\ G(w) = 0 \\ \text{the field generated by a root of } G(x) \text{ in adjunction with } \hat{\boldsymbol{E}}_1 \end{cases}$$

Thus $\boldsymbol{K}_2 = Q(w, d, D_1, D_2, \ldots, D_n)$ is of degree n over $\boldsymbol{K}_1$ and of degree n $[\boldsymbol{K}_1{:}Q]$ over Q.

DEFINITION 1. A vector $a^{(0)}$ in $\boldsymbol{K}_2^{n-1}$ $(n \geq 2)$ with components which are functions of w is called a starting or fixed vector ; the vectors $a^{(v)} \in \boldsymbol{K}_2^{n-1}$ obtained by an algorithm from $a^{(0)}$,

(1*.5) $$a^{(v)} = (a_1^{(v)}(w),\ a_2^{(v)}(w), \ldots,\ a_{n-1}^{(v)}(w)),\quad v = 0, 1, \ldots$$

are called the current vectors ; the vectors

(1*.6) $$\begin{cases} b^{(v)} = \left(b_1^{(v)}, b_2^{(v)}, \ldots, b_{n-1}^{(v)}\right) \in \hat{\boldsymbol{E}}_1^{\,n-1} \\ \text{derived from } a^{(v)} \text{ or given by any formula are called} \\ \text{the companion vectors of the } a^{(v)}. \end{cases}$$ •

We shall introduce the notation

(1*.7) $$a_0^{(v)},\ b_0^{(v)} = 1,\quad v = 0, 1, \ldots$$

DEFINITION 2. The (BGEA) of a starting vector $a^{(0)}$ is defined by the recurrence formula

(1*.8) $$a^{(v+1)} = (a_1^{(v)} - b_1^{(v)})^{-1}(a_2^{(v)} - b_2^{(v)}, \ldots,\ a_{n-1}^{(v)} - b_{n-1}^{(v)},\ 1)$$

$$a_1^{(v)} - b_1^{(v)} \bullet 0;\quad v = 0, 1, \ldots$$ •

From (1*.8) we obtain

(1*.9) $$a_i^{(v)} = b_i^{(v)} + \frac{a_{i-1}^{(v+1)}}{a_{n-1}^{(v+1)}};\quad i = 1, \ldots, n-1;\quad v = 0, 1, \ldots$$

We define the matrix entries

(1*.10) $$\begin{cases} A_i^{(j)} = \delta_j^i;\ i, j = 0, 1, \ldots, n-1,\ \delta_j^i \text{ the Kronecker delta}; \\ A_i^{(v+n)} = \sum_{j=0}^{n-1} b_j^{(v)} A_i^{(v+j)},\ i = 0, 1, \ldots, n-1;\ v = 0, 1, \ldots \end{cases}$$

From (1*.10) we obtain easily by induction

(1*.11) $$\begin{vmatrix} A_0^{(v)} & A_0^{(v+1)} & \Lambda & A_0^{(v+n-1)} \\ A_1^{(v)} & A_1^{(v+1)} & \Lambda & A_1^{(v+n-1)} \\ \Lambda & \Lambda & \Lambda & \Lambda \\ A_{n-1}^{(v)} & A_{n-1}^{(v+1)} & \Lambda & A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)}$$

for $v = 0, 1, \ldots$

We obtain formula

$$(1^*.12) \qquad a_i^{(0)} = \frac{\sum_{j=0}^{n-1} a_j^{(v)} A_i^{(v+j)}}{\sum_{j=0}^{n-1} a_j^{(v)} A_0^{(v+j)}} \;, \quad i = 1,\ldots, n-1\;; \quad v = 0, 1,\ldots;$$

$(1^*.12)$ is correct for $v = 0$, giving on the right side,

$$(0 + 0 + \cdots + a_i^{(0)} A_i^{(i)} + 0 + \cdots + 0) \,/\, A_0^{(0)} = a_i^{(0)}\,.$$

Then substitute inductively on the right the values of $a_j^{(v)}$ from $(1^*.10)$, $(1^*.9)$.

We shall need the formula

$$(1^*.13) \qquad \prod_{j=1}^{v} a_{n-1}^{(j)} = \sum_{i=0}^{n-1} a_i^{(v)} A_0^{(v+i)} \;, \quad v = 1, 2,\ldots$$

Proof by induction. $(1^*.13)$ is correct for $v = 1$. Then proceed as in the proof of $(1^*.12)$.

We shall need the basic formula

$$(1^*.14) \qquad \begin{vmatrix} 1 & A_0^{(v+1)} & A_0^{(v+2)} & \Lambda & A_0^{(v+n-1)} \\ a_1^{(0)} & A_1^{(v+1)} & A_1^{(v+2)} & \Lambda & A_1^{(v+n-1)} \\ M & M & M & & M \\ a_{n-1}^{(0)} & A_{n-1}^{(v+1)} & A_{n-1}^{(v+2)} & \Lambda & A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)} \left(\sum_{j=0}^{n-1} a_j^{(v)} A_0^{(v+j)} \right)^{-1} .$$

Proof. Start with $(1^*.11)$. Multiply both sides by $\sum_{j=0}^{n-1} a_j^{(v)} A_0^{(v+j)}$, add to the first column the $a_j^{(v)}$ multiple of the j+1 st column. The result is obtained by induction.

2^*. A periodic (BGEA) - notations. In this section we begin preparation for the periodicity of the most general $a^{(0)} \in \boldsymbol{K}_2$. This $a^{(0)}$ will be specified later when we approach the central theorem. The choice of $a^{(0)}$ will result from specification of the numbers $d, D_1, D_2,\ldots, D_n \in S_{n+1}$ and the function $G(x)$ from $(1^*.3)$. We first choose

$$(2^*.1) \qquad G_F(x) = -\,d + \prod_{j=1}^{n} \left(x - D_j\right), \quad n \geq 2,$$

where $G_F(x)$ is irreductible over $\boldsymbol{K}_1$.

$$(2^*.2) \qquad \begin{cases} G_F(w_i) = 0,\ i = 1,\ldots,n;\ \text{Let } w_1, w_2,\ldots, w_n \text{ be the roots} \\ \text{of } G_F(x) \text{ is an algebraic closure of } \hat{\boldsymbol{E}}_2(w_1,\ldots,w_n). \\ w \in (w_1, w_2,\ldots, w_n) \end{cases}$$

Thus w is chosen to be one fixed root of the n-roots of G_F. We introduce the notation

$$(2^*.3)\qquad \begin{cases} f_{i,i} = f_i = w - D_i,\ i = 1,2,...,n; \\ f_{i,k} = \prod_{j=1}^{k}(w - D_j);\ 1 \le i \le k \le n; \end{cases}$$

In $\boldsymbol{K}_2$ we have of course

$$(2^*.4)\qquad f_{i,k} = f_{i,k}(w);\ \ 1 \le i \le k \le n;$$

and need not add $f_{i,k} = f(w, D_1,.., D_k)$. We have, from $(2^*.1)$ - $(2^*.3)$

$$(2^*.5)\qquad \prod_{j=1}^{n}(w - D_j) = f_{1,n} = f_{1,n}(w) = d.$$

The following operations will be useful.

$$(2^*.6)\qquad \begin{cases} \dfrac{1}{f_{i,k}(w)} = \dfrac{1}{f_{i,k}} = \dfrac{f_{1,i-1}f_{k+1,n}}{d};\ i \ge 2,\ k \le n-1, \\ \dfrac{1}{f_{1,k}} = \dfrac{f_{k+1,n}}{d};\ \ k \le n-1,\ \ \dfrac{1}{f_{i,n}} = \dfrac{f_{1,i-1}}{d};\ \ 2 \le i \le n, \end{cases}$$

For the fixed vector $a^{(0)}$ we now choose

$$(2^*.7)\qquad a^{(0)} = (f_{1,n-1}, f_{1,n-2},.., f_{1,2}, f_{2,2}).$$

We shall in the sequel conduct the (BGEA) of $a^{(0)}$ from $(2^*.7)$, getting the current vectors $a^{(v)}$, v = 1, 2,… The companion vectors of the current vectors are derived from the current vectors by the formula, remembering $(2^*.4)$;

$$(2^*.8)\qquad \begin{cases} b_s^{(v)} = a_s^{(v)}(D_1);\ s = 1,...,n-1; \\ b^{(v)} \in \hat{\boldsymbol{E}}_1^{n-1},\ \ v = 0,1,.... \end{cases}$$

The choice of $D_1 \in S_{n+1}$ for the derivation of $b^{(v)}$ from $a^{(v)}$ is, as we shall later see, completely arbitrary. Any $D_i \in S_{n+1}$, $i = 1,..,n$ would do.

The reader should pay priority attention to the formula

$$(2^*.9)\qquad \begin{cases} b_s^{(0)} = f_{1,n\text{-}s}(D_1) = \left[(w - D_1)\wedge (w - D_{n\text{-}s})\right]_{w=D_1} = 0, \\ s = 1,...,n-1. \end{cases}$$

We shall illustrate the first step in the (BGEA) of $a^{(0)}$, working out all the necessary details of $(1^*.8)$. In the sequel the current vectors' result will sometimes be enumerated directly without going into details. We obtain from $(2^*.7)$, in view of $(2^*.9)$

$$(2^*.10)\qquad \begin{cases} b_i^{(0)} = f_{1,n\text{-}i}(D_1) = 0,\ i = 1,2,...,n\text{-}2; \\ b_{n\text{-}1}^{(0)} = f_{2,2}(D_1) = f_2(D_1) = D_1 - D_2\,. \end{cases}$$

Thus

$$b^{(0)} = (0, 0,.., 0, D_1\text{-}D_2).$$

$$a_1^{(0)} - b_1^{(0)} = f_{1,n-1} - 0 = f_{1,n-1},$$

and by (2*.6)

$$(a_1^{(0)} - b_1^{(0)})^{-1} = \frac{1}{f_{1,n-1}} = \frac{f_{n,n}}{d} = \frac{f_n}{d},$$

$$a_{n-1}^{(0)} - b_{n-1}^{(0)} = f_{2,2} - (D_1 - D_2) = w - D_2 - (D_1 - D_2) = w - D_1 = f_1,$$

$$a_i^{(0)} - b_i^{(0)} = a_i^{(0)} - 0 = a_i = f_{1,n-1}, \quad i = 2,..,n-2$$

and from (1*.8) ;

(2*.11) $$a^{(1)} = d^{-1} f_n (f_{1,n-2}, f_{1,n-3},.., f_{1,2}, f_1, 1),$$

$$a^{(1)} = (d^{-1} f_{1,n-2} f_n, d^{-1} f_{1,n-3} f_n,.., d^{-1} f_1 f_n, d^{-1} f_n).$$

3*. The first fugue of the (BGEA) of $a^{(0)}$.

DEFINITION 3. A sequence of $n-1$ current vectors (including) the fixed vector), viz.

(3*.1) $$a^{((n-1)v+j)}, \quad v = 0, 1,...; \; j = 0, 1,.., n-2$$

is called the $v+1$ st fugue (of the current vector) of the (BGEA) of $a^{(0)}$; the sequence of the corresponding companion vectors - the $v+1$ st fugue of the companion vectors of the (BGEA) of $a^{(0)}$, is

(3*.2) $$b^{(n-1)v+j}, \quad v = 0, 1,...; \; j = 0, 1,.., n-2.$$ •

From (2*.11) we obtain

(3*.3) $$b^{(1)} = (0, 0,.., 0, d^{-1}(D_1\text{-}D_2)),$$

$$a_1^{(1)} - b_1^{(1)} = d^{-1} f_{1,n-2} f_n - 0 \; ; \; (a_1^{(1)} - b_1^{(1)})^{-1} = f_{n-1},$$

$$a_{n-1}^{(1)} - b_{n-1}^{(1)} = d^{-1} f_n - d^{-1}(D_1 - D_n) = d^{-1} f_1,$$

$$a^{(2)} = f_{n-1}(d^{-1} f_{1,n-3} f_n, .., d^{-1} f_{1,2} f_n, d^{-1} f_1 f_n, d^{-1} f_1, 1),$$

(3*.4) $$a^{(2)} = (d^{-1} f_{1,n-3} f_{n-1,n}, .., d^{-1} f_{1,2} f_{n-1,n}, d^{-1} f_1 f_{n-1,n}, d^{-1} f_1 f_{n-1}, f_{n-1}).$$

We can now prove the important

LEMMA 1. The $i+1$ st current vector of the first fugue has the form

(3*.5) $$\begin{cases} a^{(i)} = (d^{-1} f_{1,n-i-1} f_{n-i+1,n}, d^{-1} f_{1,n-i-2} f_{n-i+1,n}, \dots, \\ \qquad d^{-1} f_1 f_{n-i+1,n}, d^{-1} f_1 f_{n-i+1,n-1}, f_1 f_{n-i+1,n-2}, \dots, f_1 f_{n-i+1}, f_{n-i+1}) \\ i = 2, \dots, n-2; \quad n \geq 4. \end{cases}$$

Proof. ($3^*.5$) is correct for $i = 2$, as can be verified from ($3^*.4$). By the method applied in calculating $a^{(1)}$, $a^{(2)}$, the reader will now have no difficulty in proving Lemma 1 by induction. The special cases $n = 2, 3$ will be observed separately because of their importance.

$$a^{(0)} = w - D_2 \,;\; b^{(0)} = D_1 - D_2$$

$$a^{(1)} = d^{-1} (w - D_2) \,; \quad b^{(1)} = d^{-1} (D_1 - D_2)$$

$$a^{(2)} = w - D_2 = a^{(0)} .$$

Here we have a purely periodic continued fraction representation.

$$(w - D_1)(w - D_2) - d = 0 \,;\; w - D_2 = \left\lfloor \overline{D_1 - D_2} \right\rfloor, \quad d = 1.$$

$$w - D_2 = \left\lfloor \overline{(D_1 - D_2)\, d^{-1}, (D_1 - D_2)} \right\rfloor, \quad d > 1 \,;\; d \mid D_1 - D_2 .$$

For $n = 3$, we obtain, $d \neq 1$.

$a^{(0)} = ((w - D_1)(w - D_2), w - D_2) \,;$ $b^{(0)} = (0, D_1 - D_2) \,;$

$a^{(1)} = (d^{-1} (w - D_1)(w - D_3), d^{-1} (w - D_3)) \,;$ $b^{(1)} = (0, d^{-1} (D_1 - D_3)) \,;$

$a^{(2)} = (d^{-1} (w - D_1)(w - D_2), w - D_2) \,;$ $b^{(2)} = (0, D_1 - D_2) \,;$

$a^{(3)} = ((w - D_1)(w - D_3), w - D_3) \,;$ $b^{(3)} = (0, D_1 - D_3) \,;$

$a^{(4)} = (d^{-1} (w - D_1)(w - D_2), d^{-1} (w - D_2)) \,;$ $b^{(4)} = (0, d^{-1} (D_1 - D_2)) \,;$

$a^{(5)} = (d^{-1} (w - D_1)(w - D_3), w - D_3) \,;$ $b^{(5)} = (0, D_1 - D_3) \,;$

$a^{(6)} = ((w - D_1)(w - D_2), w - D_2) = a^{(0)} .$

The (BGEA) of $a^{(0)}$ is purely periodic, and the length of its primitive period is $m = 6$.

For $d = 1$ we obtain

$a^{(0)} = ((w - D_1)(w - D_2), w - D_2) \,;$ $b^{(0)} = (0, D_1 - D_2) \,;$

$a^{(1)} = ((w - D_1)(w - D_3), w - D_3) \,;$ $b^{(1)} = (0, D_1 - D_3) .$

$a^{(2)} = a^{(0)} \,;$ $(w - D_1)(w - D_2)(w - D_3) - d = 0,$ generally.

4^*. The first fugue of the (BGEA) of $a^{(0)}$, continued.

From ($2^*.7$), ($2^*.11$) and ($3^*.5$) we have completed the calculations of the first fugue of the (BGEA) of $a^{(0)}$, viz.

$$\langle a^{(0)}, a^{(1)}, \ldots, a^{(i)}, \ldots, a^{(n-2)} \rangle, \quad i = 2, \ldots, n-2$$

altogether the fixed vector $a^{(0)}$, followed by $n - 2$ current vectors. We obtain for the $n - 1$ st vector, substituting $i = n - 2$ in ($3^*.5$),

($4^*.1$) $$a^{(n-2)} = (d^{-1} f_{1,1} f_{3,n}, d^{-1} f_1 f_{3,n-1}, f_1 f_{3,n-2}, \ldots, f_1 f_3, f_3) .$$

From $(4^*.1)$ we obtain, calculating (in detail because of its essential pattern) the next current vector $a^{(n-1)}$. For this purpose we have

$(4^*.2)$ $$b^{(n-2)} = (0, 0, 0, \ldots, 0, D_1 - D_3)$$

and from $(4^*.1)$, $(4^*.2)$

$(4^*.3)$ $$\begin{cases} a_1^{(n-2)} - b_1^{(n-2)} = d^{-1} f_{1,1} f_{3,n}; \ \left(a_1^{(n-2)} - b_1^{(n-2)}\right)^{-1} = f_2; \\ a_{n-1}^{(n-2)} - b_{n-1}^{(n-2)} = w - D_3 - (D_1 - D_3) = w - D_1 = f_1. \end{cases}$$

From $(4^*.1)$ - $(4^*.3)$ we obtain, by virtue of Definition 1,

$(4^*.4)$ $$a^{(n-1)} = f_2 (d^{-1} f_1 f_{3,n-1}, f_1 f_{3,n-2}, \ldots, f_1 f_3, f_1, 1)$$

$$a^{n-1} = (d^{-1} f_{1,n-1}, f_{1,n-2}, \ldots, f_{1,2}, f_2).$$

$(4^*.4)$ is important. If we compare the latter with $(2^*.7)$ we obtain

$(4^*.5)$ $$a^{(0)}{}_{(d)} = a^{(n-1)}$$

which means to say that $a^{(0)} = a^{(n-1)}$ in the case $d = 1$. We have thus obtained the interesting

THEOREM 1. The (BGEA) of the fixed vector $a^{(0)}$ from $(2^*.7)$, with the notations $(2^*.3)$ and $(2^*.5)$, is purely periodic in the case $d = 1$, and the length of the primitive period equals $m = n - 1$.

The latter has the form

$$\langle a^{(0)}, a^{(1)}, \ldots, a^{(i)}, \ldots, a^{(n-2)} \rangle, \quad i = 2, \ldots, n-2.$$

$a^{(0)}$ from $(2^*.7)$, $a^{(1)}$ from $(2^*.11)$, $a^{(i)}$ from $(3^*.5)$ $(i = 2, \ldots, n-2)$, substituting in these formulas $d = 1$. The corresponding companion vectors have the form :

$(4^*.6)$ $$\begin{cases} b^{(0)} = (0,0,\ldots,0, D_1 - D_2), \\ b^{(1)} = (0,0,\ldots,0, D_1 - D_n), \ldots, \\ b^{(i)} = (0,0,\ldots,0, D_1 - D_{n-i+1}), \quad i = 2, \ldots, n-2. \end{cases}$$ •

The reader is reminded that while $d = 1$ in Theorem 1, the other n elements $D_1, \ldots, D_n$ of the set S_{n+1} may still be any algebraic numbers. It is surprising that in the case $d = 1$ the (BGEA) of the fixed vector $a^{(0)}$ from $(2^*.7)$ indeed is purely periodic with the astonishingly short primitive period $m = n - 1$. In the case $d \neq 1$, the (BGEA) of the same $a^{(0)}$ is also purely periodic, but the length of its primitive period equals $m = n(n-1)$. This will be proved in the next sections. But already, we have learned from the previous section, that in the case $n = 2$ the (BGEA) of $a^{(0)}$ from $(2^*.7)$ has primitive period of length $m = 2 = 2(2-1)$ when $d \neq 1$, and $m = 1 =$

2–1 when $d = 1$; and in the case $n = 3$, $m = 6 = 3(3-2)$ when $d \neq 1$, and $m = 2 = 3-1$ when $d = 1$.

5*. Periodicity of the (BGEA) of $a^{(0)}$ - completed.

We shall use a new manner of writing the vectors of the primitive period of the (BGEA) of $a^{(0)}$ from Theorem 1 (with $d = 1$) and introduce

(5*.1) $$g^{(i)} = \left(g_1^{(i)}, g_2^{(i)}, ,..., g_{n-1}^{(i)}\right), \quad i = 0, 1,.., n-2 ; \quad n \geq 4.$$

It will be useful, for later purposes, to write out in full the values of $g_j^{(i)}$ from (2*.7), (2*.11), (3*.5) where $d = 1$,

(5*.2) $$\begin{cases} g^{(0)} = \left(g_1^{(0)}, g_2^{(0)}, ..., g_{n-1}^{(0)}\right) = \left(f_{1,n-1}, f_{1,n-2}, ..., f_{1,2}, f_2\right); \\ g^{(1)} = \left(g_1^{(1)}, g_2^{(1)}, ..., g_{n-1}^{(1)}\right) = \left(f_{1,n-2} f_n, f_{1,n-3} f_n, ..., f_{1,2} f_n, f_1 f_n, f_n\right); \\ g^{(i)} = \left(g_1^{(i)}, g_2^{(i)}, ..., g_{n-1}^{(i)}\right) = (f_{1,n-i-1} f_{n-i+1,n}, f_{1,n-i-2} f_{n-i+1,n}, ..., \\ \qquad f_{1,2} f_{n-i+1,n}, f_1 f_{n-i+1,n}, f_1 f_{n-i+1,n-1}, ..., f_1 f_{n-i+1}, f_{n-i+1}), \\ i = 2, 3, ..., n-2. \end{cases}$$

With the notation (5*.2) we shall now write out in full a few of the fugues of the (BGEA) of $a^{(0)}$ from (2*.7) with $d \neq 1$.

(5*.3) $$\begin{cases} a^{(0)} = \left(g_1^{(0)}, g_2^{(0)}, ..., g_{n-1}^{(0)}\right), \\ a^{(1)} = \left(d^{-1} g_1^{(1)}, d^{-1} g_2^{(1)}, ..., d^{-1} g_{n-1}^{(1)}\right), \\ a^{(2)} = \left(d^{-1} g_1^{(2)}, d^{-1} g_2^{(2)}, ..., d^{-1} g_{n-1}^{(2)}\right), \\ \vdots \\ a^{(n-2)} = \left(d^{-1} g_1^{(n-2)}, d^{-1} g_2^{(n-2)}, g_3^{(n-2)}, .., d^{-1} g_{n-1}^{(n-2)}\right). \end{cases}$$

(5*.3) is the first fugue.

(5*.4) $$\begin{cases} a^{(1(n-1)+0)} = \left(d^{-1} g_1^{(0)}, g_2^{(0)}, g_3^{(0)}, ..., g_{n-1}^{(0)}\right), \\ a^{(1(n-1)+1)} = \left(g_1^{(1)}, g_2^{(1)}, ..., g_{n-1}^{(1)}\right), \\ a^{(1(n-1)+2)} = \left(d^{-1} g_1^{(2)}, d^{-1} g_2^{(2)}, ..., d^{-1} g_{n-1}^{(2)}\right), \\ a^{(1(n-1)+3)} = \left(d^{-1} g_1^{(3)}, d^{-1} g_2^{(3)}, ..., d^{-1} g_{n-2}^{(3)}, g_{n-1}^{(3)}\right), \\ \vdots \\ a^{(1(n-1)+n-2)} = \left(d^{-1} g_1^{(n-2)}, d^{-1} g_2^{(n-2)}, d^{-1} g_3^{(n-2)}, g_4^{(n-2)}, .., g_{n-1}^{(n-2)}\right). \end{cases}$$

(5*.4) is the second fugue.

$$(5^*.5)\qquad \begin{cases} a^{(2(n-1)+0)} = \left(d^{-1}g_1^{(0)}, d^{-1}g_2^{(0)}, g_3^{(0)}, \ldots, g_{n-1}^{(0)}\right), \\ a^{(2(n-1)+1)} = \left(d^{-1}g_1^{(1)}, g_2^{(1)}, \ldots, g_{n-1}^{(1)}\right), \\ a^{(2(n-1)+2)} = \left(g_1^{(2)}, g_2^{(2)}, \ldots, d^{-1}g_{n-1}^{(2)}\right), \\ a^{(2(n-1)+3)} = \left(d^{-1}g_1^{(3)}, d^{-1}g_2^{(3)}, \ldots, d^{-1}g_{n-1}^{(3)}\right), \\ a^{(2(n-1)+4)} = \left(d^{-1}g_1^{(4)}, d^{-1}g_2^{(4)}, \ldots, d^{-1}g_{n-2}^{(4)}, g_{n-1}^{(4)}\right), \\ \vdots \\ a^{(2(n-1)+n-2)} = \left(d^{-1}g_1^{(n-2)}, \ldots, d^{-1}g_4^{(n-2)}, g_5^{(n-2)}, \ldots, g_{n-1}^{(n-2)}\right). \end{cases}$$

$(5^*.5)$ is the third fugue.

The reader will now have no difficulty in proving

LEMMA 2. The factors d^{-1} appearing before the $g_i^{(j)}$ in the current vectors of the (BGEA) of $a^{(0)}$ from $(2^*.7)$ with $d \neq 1$ appear subsequently one after the other starting with the first component of the vector; their frequency of occurrence in a vector is k, $0 \le k \le n-1$; if in a vector the frequency of d^{-1} is k, then in the next vector it is $k-1$; if its frequency in one vector is zero, then in the next vector is $n-1$.

For the j th vector of the $i+1$ st fugue we shall use the notation

$$(5^*.6)\qquad a^{(i(n-1)+j)} = (k)\ g^{(j)}$$

where (k) denotes the frequency of the factor d^{-1} in this vector $g^{(j)}$. Of course, $(5^*.6)$ does not say how to state explicitly any such vector $a^{(i(n-1)+j)}$. For this purpose a functional relation between i, j and k is necessary and most important. This is :

$$(5^*.7)\qquad \begin{cases} i - j \equiv k(n), & k = 0,1,\ldots,n-1; \\ \text{For } k < 0, \text{ take } n-k. & j = 0,1,\ldots,n-2. \end{cases}$$

Then proof of $(5^*.7)$ is entirely based on Lemma 2.

If in a certain fugue we have the vector $a^{i(n-1)+j}$, then in the next vector we have $j+1$, and since k decreases by one in this next vector, it becomes $k-1$, so that $i-(j+1) \equiv k-1(n)$ or $i-j \equiv k(n)$, as should be. If we look for the next fugue with the same row-vector j, viz. $a^{(i+1)(n+1)+j}$, then the frequency of k for the same j is one greater, namely $k+1$, so that we have $(i+1) - j \equiv k+1(n)$ or again $i - j \equiv k(n)$. This proves $(5^*.7)$ by induction, since it is correct for $i = j = = k = 0$.

We are now able to write down any current vector in the (BGEA) of $a^{(0)}$. For example, let $n = 9$, $m = 9 \cdot 8 = 72$, and we want to find $a^{(61)} = a^{7(9-1)+5}$; $i = 7$, $j = 5$, $7-5 = 2 = k$. Hence

$$a^{(61)} = \left(d^{-1}g_1^{(5)}, d^{-1}g_2^{(5)}, g_3^{(5)}, \ldots, g_8^{(5)}\right).$$

We now ask the decisive question ; Can (BGEA) of $a^{(0)}$ become purely periodic, and what is the length m of the primitive period ? (Every periodic (BGEA) can be transformed into a purely periodic (BGEA)). In such a case we must have

$(5^*.8)$ $\qquad j = k = 0$, as in $a^{(0)}$.

$(5^*.7)$, $(5^*.8)$ result in choosing min. i,

$(5^*.9)$ $\qquad i \equiv 0\ (n), \quad i = n.$

We have obtained :

THEOREM 2. The (BGEA) of the fixed vector $a^{(0)}$ from $(2^*.7)$ with the notation $(2^*.3)$, $(2^*.5)$ and $d \neq 1$ is purely periodic and the length of its primitive period is $m = n(n-1)$, consisting of n fugues. •

The length of the primitive period of the (BGEA) is indeed very large .

6^*. Irreducibility and roots of polynomials.

Bernstein has proved the following result which we state here with some slight alterations :

THEOREM 3. Let

$$(6^*.1)\qquad \begin{cases} P(x) = x^n + \left(\sum_{j=1}^{n-1} k_j x^{n-j}\right) - d, \\ d, k_1, \ldots, k_{n-1} \in Z;\ d \mid k_j \quad (j = 1, \ldots, n-1), \\ |k_{n-1}| \geq 2|d|(2 + B);\ \ B = 1 + \sum_{j=1}^{n-2} |k_j|. \end{cases}$$

Then P(x) is irreducible and has at least one real root. •

If d is square free and $|d| > 1$, then irreducibility of P(x) follows from Eisenstein's Criterion. But this would then exclude $|d| = 1$. Bernstein's theorem is valid also for any algebraic integers $d, k_1, \ldots, k_{n-1}$, $d \mid k_{n-1}$. This we could well use, but it will take us too long to prove it. We introduce a polynomial which will be central to our investigation on units, viz.

(6*.2)
$$\begin{cases} T(x) = -d + \prod_{i=1}^{k} (x^{s_i} - D_i^{s_i}), \\ d, D_i \in Z;\ d \mid D_i \quad i = 1,2,\dots,k; \\ s_i \ge 1;\ \text{if all } s_i = 1, \text{ then } k \ge 2; \\ 0 < D_1 < D_2 < \dots < D_k. \end{cases}$$

We prove

LEMMA 3. $T(x)$ is irreducible in infinitely many cases and has, in these cases, at least one real root. •

We rearrange (6*.2) :

(6*.3)
$$\begin{cases} T(y) = -d + \prod_{i=1}^{k} [(y + D_k)^{s_i} - D_i^{s_i}], \\ y = x - D_k. \end{cases}$$

From (6*.3)

(6*.4)
$$\begin{cases} T(y) = -d + \left(y^{s_k} + \binom{s_k}{1} D_k y^{s_k - 1} + \dots + \binom{s_k}{s_k - 1} D_k^{s_k - 1} y \right) \cdot \\ \cdot \prod_{i=1}^{k-1} [(y + D_k)^{s_i} - D_i^{s_i}] \\ = y^n + t_1 y^{n-1} + \dots + t_{n-1} y - d, \\ d \mid t_i;\ t_{n-1} = \binom{s_k}{s_k - 1} D_k^{s_k - 1} \cdot \prod_{i=1}^{k-1} (D_k^{s_i} - D_i^{s_i}), \\ n = s_1 + s_2 + \dots + s_k;\quad i = 1,\dots,n-1 \text{ in } t_i, \end{cases}$$

where $d \mid t_i$ since all t_i are polynomials in $D_1, D_2, \dots, D_k$ with rational coefficients ; further t_{n-1} contains the highest power of D_k, viz. $s_k - 1 + s_1 + + s_2 + \dots + s_{k-1} = n - 1$. Hence we can choose sufficiently large D_k so that

$$|k_{n-1}| \ge 2|d|(2 + B_1);\quad B_1 = 1 + \sum_{j=1}^{n-2} |t_j|.$$

Thus the polynomial $T(y)$, hence also $T(x)$, satisfies the conditions of the polynomial $P(x)$, and Theorem 3, so that $T(x)$ is irreducibile and has at least one real root in infinitely many cases.

In case $s_k = n$, $T(x)$ becomes $T(x) = x^n - D^n - d,\ d \mid D,\ n \ge 2;\ D \ge 1$.

$$T(y) = y^n + \sum_{j=1}^{n-1} \binom{n}{j} D^j y^{n-j} - d; \quad d, D \in Z,$$

and we have to choose

$$\binom{n}{1} D^{n-1} \geq 2|d| \left[3 + \binom{n}{1} D + \binom{n}{2} D^2 + \dots + \binom{n}{n-2} D^{n-2} \right].$$

It suffices to choose $D \geq 2^{n+1}|d|$; $n \geq 2$. Concluding, we shall write $T(x)$ in the form of $(2^*.1)$.

$$(6^*.5) \qquad \begin{cases} G_T(x) = -d + \prod_{i=1}^{k} \prod_{j=1}^{s_i} \left(x - \rho_j^{j-1} D_j \right) \\ \rho_j = \exp \dfrac{2\pi\sqrt{-1}}{s_i}. \text{ (We write } \sqrt{-1} \text{ to avoid} \\ \text{confusion with the index i.)} \end{cases}$$

$$(6^*.6) \qquad \begin{cases} \{D_1, \rho_1 D_1, \dots, \rho_1^{s_1-1} D_1, D_2, \rho_2 D_2, \dots, \rho_2^{s_2-1} D_2, \dots, D_k, \rho_k D_k, \dots, \rho_k^{s_k-1} D_k\} \\ = \{\overline{D_1}, \overline{D_2}, \dots, \overline{D_n}\} \quad \text{for useful notation.} \end{cases}$$

Let

$$(6^*.7) \qquad \begin{cases} (s_1, s_2, \dots, s_k) = s; \ \bar{\rho} = \exp \dfrac{2\pi\sqrt{-1}}{s}. \\ G_T(\overline{w}) = 0; \ \overline{w} \text{ real}; \ Q(\overline{D_1}, \overline{D_2}, \dots, \overline{D_k}, \overline{w}) = Q(\bar{\rho}, w). \end{cases}$$

$$(6^*.8) \qquad G_T(x) = -d + \prod_{j=1}^{n} (x - \overline{D_j}); \quad -d + \prod_{j=1}^{n} (\overline{w} - \overline{D_j}) = 0.$$

With this we conclude the proof of Theorem 2 which proves the necessary condition for the periodicity of (BGEA).

Thus, we have achieved our main goal, namely, we have constructed the (BGEA) algorithm by which a vector $a^{(0)} \in Q(w)^{n-1}$ of dimension (n-1), $w = \sqrt[n]{D^n + d}$, $d \in Z$, $D \in N$, $d \mid D$ becomes periodic without the restriction $D \geq (n-2)d$ for $d > 0$ or $D \geq 2(n-1)|d|$ for negative d, $n \geq 3$.

It is important to note for later purpose that the $b_i^{(v)}$ of the (BGEA) algorithm we have constructed are all algebraic integers.

We conclude this section by showing that in the case $n = 2$, (BGEA) coincides with continued fractions, thus $n = 2$ in (BGEA) is the (EA).

(6*.9)
$$\begin{cases} w = \sqrt{D^2 + d},\ d \mid D,\ d \in Z \text{ and } D \in N \\ a^{(0)} = w + D. \end{cases}$$

In this case (as in any other case of real quadratic irrational) the (JPA) becomes the (EA) which gives the expansion of $a^{(0)}$ as a simple continued periodic fraction.

By the definition of (JPA) we have here

(6*.10)
$$\begin{cases} a^{(v+1)} = \dfrac{1}{a^{(v)} - b^{(v)}} \cdot 1 = \dfrac{1}{a^{(v)} - b^{(v)}} \\ a^{(v)} = b^{(v)} + \dfrac{1}{a^{(v+1)}};\ b^{(v)} = [a^{(v)}];\ v = 1,2,... \\ a^{(0)} = b^{(0)} + \cfrac{1}{b^{(1)} + \cfrac{1}{b^{(2)} + \Lambda}} \end{cases}$$

$a^{(0)} = (w + D)$

$[w] = D$, $b^{(0)} = 2D$

$a^{(1)} = \dfrac{1}{a^{(0)} - b^{(0)}} = \dfrac{1}{w - D} = \dfrac{w + D}{d}$

$b^{(1)} = \dfrac{2D}{d}$

$a^{(2)} = \dfrac{1}{a^{(1)} - b^{(1)}} = \dfrac{d}{w - D} = w + D$.

Using (BGEA) in the case $n = 2$ for $b^{(i)}$ as in (4*.6)

$b^{(i)} = (0, 0,.., 0, D_1 - D_{n-i+1})$. $i = 2,..,n-2$;

we have, using

$w^2 - D^2 = d$, $(w - D)(w + D) = d$, $-D = -D_1$; $+D = -D_2$

$(w - D_1)(w - D_2) = d$.

$b^{(v)} = a^{(v)}(D_1)$

$a^{(0)} = w + D = w - D_2$, $b^{(0)} = D_1 - D_2$

$a^{(0)} - b^{(0)} = w - D_1$

$a^{(1)} = \dfrac{1}{w - D_1} = \dfrac{w - D_2}{d}$, $b^{(1)} = \dfrac{D_1 - D_2}{d}$

$a^{(1)} - b^{(1)} = \dfrac{w - D_2}{d}$

$$a^{(2)} = \frac{1}{a^{(1)} - b^{(1)}} = \frac{d}{w - D_1} = w + D_2 = a^{(0)}, \quad b^{(2)} = D_1 - D_2.$$

Section 2. The proof of the sufficient condition for the periodicity of (BGEA).

In 1980, in her PHD disertation with Dr. Jurgen Schmidt as her PhD advisor, Baica proved that $d \mid D$ is a necessary condition for (BGEA) periodicity. Hasse, who was the author's PhD disertation advisor, asked Baica to do something with this restriction, that is, to show either that it can be eliminated and the Fermat last theorem is false or it can not be eliminated and then it proves Fermat last theorem by generalization from quadratics. This will mean to prove that $d \mid D$ is also the sufficient condition in proving the Euler direction for the periodicity of (BGEA) making this algorithm to be the General Euclidean Algorithm which is <u>not always</u> periodic. That is not every n-degree irrational makes (BGEA) always periodic.

Hilbert realized that (EA) is a very powerful algorithm because it is always periodic. As a result of its always periodicity many important problems in quadratics or E^2 were completely proved from its periodicity.

The same famous problems in n-dimensions or E^n remained open questions in the algebraic number theory. All of these open questions for $n > 2$ caused Hilbert to ask for the invention of a universal algorithm as powerful as (EA) in quadratics ($n = 2$) in order to solve all of same problems for higher dimensions (n) from the periodicity of this universal algorithm. This Hilbert "Zahlbericht" is known as Hilbert's 10-th problem.

Logicians proved that this Hilbert's dreamed algorithm to be always periodic does not exist.

In his 10-th problem Hilbert asked for the General Euclidean Algorithm (GEA) and for the n-dimensional equivalent of (ELT) from quadratics to prove its always periodicity.

We use the result proved by the logicians (not an explicite proof) of Hilbert's 10^{th} problem to prove that $d \mid D$ is also a sufficient condition in proving (BGEA) restricted periodicity. By logic it was proved that Hilbert's dreamed periodic algorithm does not exists to be always periodic.

If $d \mid D$ in the periodicity of (BGEA) could be eliminated then it is in contradiction with Hilbert's 10-th problem proved by the logicians, and therefore the restriction $d \mid D$ can not be eliminated in proving (BGEA) periodic and as such (BGEA) now is proved to be restrictiv periodic. It is true that if $d \mid D$, then (BGEA) is not periodic since otherwise it will contradict Hilbert's 10-th problem.

This completed the proof for the restricted periodicity.

Since (BGEA) is of the same cut or prototype as (EA), then (BGEA) is the only General Euclidean Algorithm.

Section 3. (BGEA) an n-dimensional equivalent of Euler – Lagrange Theorem (ELT) from quadratics.

In 1737, Euler proved that every real quadratic irrational can be represented by an infinite simple periodic continued fraction (PSCF) or by a periodic euclidean algorithm sequence development. The converse was proved by Lagrange in 1770.

These proofs are known as the Euler – Lagrange Theorem (ELT) and proves the periodicity of the Euclidean Algorithm using simple continued fractions (PSCF) which now becomes to be another interpretation of the Euclidean Algorithm.

In 1907, Perron proved completely Lagrange direction for higher degree irrationals using his (JPA) algorithm. He showed that if the development of an n-dimensional irrational has a periodic (JPA) development then the components of the initial vector are algebraic numbers.

In 1980, Baica [1] considered for the first time the complex numbers and proved that in the periodicity of her (ACF) algorithm now (BGEA) algorithm, $d \mid D$ is a necessary condition in proving Euler's direction in the n-dimensional $(n \geq 3)$ equivalent of the Euler – Lagrange Theorem (ELT) from quadratics.

In 1995, Baica proved that $d \mid D$ for $n \geq 3$ is also a sufficient condition, and with that she proved the restricted periodicity of her Baica's General Euclidean Algorithm (BGEA) completely and as such she proved Euler direction for n-dimensional $(n \geq 3)$ completely.

This last proof makes (BGEA) the only General Euclidean Algorithm and it is the evolutionary development of the Jacobi, Perron, Hasse, Bernstein and Baica algorithms. In (BGEA) $n = 2$ becomes (EA), $n = 3$ becomes (JA), for any $n \geq 3$ of any real number is Perron (PA), (JPA) modification for reals is (HBA) and (HBA) extension over the complex numbers is (BGEA). Ultimately, (BGEA) is the General Euclidean Algorithm. Only for $n = 2$, (BGEA) coincide with (PSCF) since the periodicity of the (EA) was proved by Euler and Lagrange using (SCF), therefore every quadratic irrational and some limited number of higher degree irrationals (used by Jacobi and Perron as numerical examples in their attempt to prove the periodicity of their algorithm) have periodic (SCF) development. All of the other irrationals of degree $n \geq 3$ which makes (BGEA) periodic have a periodic (BGEA) algorithmic development.

In proving its restricted periodicity, (Euler direction) the author proved that (BGEA) restricted periodicity becomes a theorem which is the n-dimensional equivalent of the Euler – Lagrange Theorem from quadratics, Lagrange direction being proved by Perron in 1907.