CHAPTER I I I - (BGEA) AND THE SOLUTION OF ALL THE OPEN PROBLEMS IN THE ALGEBRAIC NUMBER THEORY

Section 0. Introduction.

It is known that many important and famous problems in algebraic number theory remained open for more than one hundred years and many great mathematicians spent their professional life to solve them. In this chapter we will show that (BGEA) can solve all of them.

Section 1. (BGEA) is more than the General Simple Continued Fractions Algorithm.

Many mathematicians keep calling wrongly Baica's General Euclidean Algorithm as Baica's General Continued Fractions Algorithm.

It only happens that for $n = 2$ in (BGEA) which is (EA), (BGEA) can be identified with the Continued Fractions because (ELT) proves the always periodicity of the (EA) using the Continued Fractions Algorithm. Jacobi and Perron used General Continued Fraction Algorithm but they could not prove the periodicity or the complete restricted periodicity of their algorithm except for some numerical examples.

The transformation in the Continued Fraction Algorithm is the greatest integer function. Once that Baica proved completely the restricted periodicity for (BGEA) this (BGEA) for real numbers also proves completely the restricted periodicity for (JPA) making (JPA) the General Continued Fraction Algorithm now. (HBA) is the closest (GEA) for reals and they did not use the greatest integer function as their transformation, but instead, they used the evaluation function as their transformation. Baica used the same transformation as (HBA) for the first time over the complex numbers field and with this she proved an if and only if theorem for her (BGEA) restricted periodicity. I did not claim that my algorithm (BGEA) is anything else except the General Euclidean Algorithm, which is more than the General Simple Continued Fraction Algorithm.

EXAMPLE 1. The development of any quadratic irrational W using (BGEA)

Let $a^{(0)} = W - D_2$ and $b^{(0)} = D_1 - D_2$

Where $(W - D_1)(W - D_2) - d = 0$

Then by (BGEA) $a^{(1)} = d^{-1}(W - D_2)$; $b^{(1)} = d^{-1}(D_1 - D_2)$;

$a^{(2)} = W - D_2 = a^{(0)}$.

Here we have a purely periodic continued fraction representation for $W - D_2$ where

$$(W - D_1)(W - D_2) - d = 0 ; \quad W - D_2 = [\overline{D_1 - D_2}] \text{ if } d = 1$$

$$W - D_2 = [\overline{(D_1 - D_2)d^{-1}, (D_1 - D_2)}] \; : \; d > 1 \; ; \; d \mid D_1\text{-}D_2$$

This a very simple way of construction the periodic continued fraction of $\sqrt{3}$.

For $n = 2$; $(W - 4)(W - 2) - 2 = 0$; choose $W = 3 + \sqrt{3}$; $D_1 = 4$; $D_2 = 2$; $d = 2$; $K_1 = Q$; $K_2 = Q(\sqrt{3})$;

$$a^{(0)} = W - 2 = [\overline{\frac{(4-2)}{2}, (4-2)}]$$

$$\sqrt{3} + 3 - 2 = [\overline{1,2}] \; ; \; \sqrt{3} + 1 = [\overline{1,2}] \text{ and } \sqrt{3} = [1, \overline{1,2}]$$

Similarly for $\sqrt{2}$

For $n = 2$; $(W - 1)(W + 1) - 1 = 0$; choose $W = 1 + \sqrt{2}$; $D_1 = 1$; $D_2 = -1$; $d = 1$; $K_1 = Q$; $K_2 = Q(\sqrt{2})$;

$$a^{(0)} = W + 1 = \sqrt{2} + 1 = [\overline{2}] \text{ and } \sqrt{2} = [1, \overline{2}]$$

EXAMPLE 2. The development of $\sqrt{2}$ and $\sqrt{3}$ by the continued fraction algorithm

For $a^{(0)} = \sqrt{3}$, $b^{(0)} = 1$

$$a^{(1)} = \frac{1}{(\sqrt{3}-1)} \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \frac{\sqrt{3}+1}{2} \; ; \quad b^{(1)} = 1$$

$$a^{(2)} = \frac{1}{\frac{\sqrt{3}+1}{2} - 1} = \frac{2}{\sqrt{3}-1} \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \sqrt{3}+1 \; ; \; b^{(2)} = 2$$

$$a^{(3)} = \frac{1}{\sqrt{3}+1-2} = \frac{1}{\sqrt{3}-1} \frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \frac{\sqrt{3}+1}{2} = a^{(1)}$$

$$\sqrt{3} = [b^{(0)}, \overline{b^{(1)}, b^{(2)}}] = [1, \overline{1,2}]$$

For $a^{(0)} = \sqrt{2}$, $b^{(0)} = 1$

$$a^{(1)} = \frac{1}{(\sqrt{2}-1)} \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{1} \; ; \quad b^{(1)} = 2$$

$$a^{(2)} = \frac{1}{\sqrt{2}+1-2} = \frac{1}{\sqrt{2}-1} \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{1} = a^{(1)}$$

$$\sqrt{2} = [1, \overline{2}]$$

Note that only for $n = 2$; (BGEA) identify with the simple continued fractions because $n = 2$ in (BGEA) is the (EA), for $n \geq 3$ the restricted periodicity of (BGEA) is not proved by continued fractions and therefore it is not a generalization of the Continued Fractions Algorithm, it is the generalization of the Euclidean Algorithm. It is true that sometimes some higher degree irrationals developed using (BGEA) when $d \mid D$ may coincide with the development using the continued fractions. There are far more irrationals which have (BGEA) periodic development than those who have periodic simple fractions development using the (JPA), and this makes (BGEA) to be more than the General Continued Fractions Algorithm.

Section 2. (BGEA) and the solution of the Hermite's problem.

In 1839, Hermite, in one of his letters to Jacobi [V], challenged Jacobi to find an algorithm to develop irrationals of any degree into periodic sequence. This is known as Hermite's problem. In other words Hermite is asking for the invention of the General Continued Fractions Algorithm, (GCFA). Let us give the (BGEA) periodic development of irrationals $n \geq 2$.

1^*. Case n = 2.

Though this case is well known from the expansion of real quadratic irrationals as simple continued fractions, we shall include it in our discussion

Let

$(1^*.1)$ $\qquad w = \sqrt{D^2 + 1}$, $D \in N$, w a quadratic irrational.

That w is irrational (for $D > 0$) is trivial. We choose the fixed vector

$(1^*.2)$ $\qquad a^{(0)} = w + D,$

since here $n - 1 = 1$. Thus $a_1^{(0)} = a_{n-1}^{(0)}$ and we shall generally denote

$(1^*.3)$ $\qquad a^{(v)} = a_v$, $v = 0, 1,\ldots;$ $a_v = a_v(\alpha)$ for all (BGEA) of $a^{(0)}$ 7.

By $(1^*.3)$ we denote

$$b^{(v)} = b_v \ , \quad v = 0, 1,\ldots;$$

For the calculations of the companion vectors we use the rule

$(1^*.4)$ $\qquad b^{(v)} = b_v = a_v(D)$, $v = 0, 1,\ldots$

and have

$(1^*.5)$ $\qquad b_0 = (w + D)_{w=D} = 2D.$

hence, by $(1^*.1)$

$(1^*.6)$ $a_1 = [(w + D) - 2D]^{-1} \cdot 1 = (w - D)^{-1} = w + D$

since $(w - D)^{-1} = w + D$ from $(w^2 - D^2) = 1$. Thus

$(1^*.7)$ $a_0 = a_1 = \ldots = a_v$, $v = 0, 1, \ldots$

and the (BGEA) of $a_0 = w + D$ is purely periodic with length of the primitive period $l = 1$. Further

$(1^*.8)$ $[a_v] = [w + D] = [w] + D = 2D = b_v$

the (BGEA) of $W + D$ coincides with the (EA) and we have, in the notation of continued fractions

$(1^*.9)$ $a_0 = w + D = [\overline{2D}]$

$(1^*.9)$ is the periodic (BGEA) development of a quadratic irrational $a_0 = w + D$.

2^*. Case n = 3.

We denote

$(2^*.1)$ $w = \sqrt[3]{D^3 + 1}$

and choose the fixed vector

$(2^*.2)$ $a^{(0)} = (w + 2D, w^2 + Dw + D^2)$

with $a^{(0)} = (a_1^{(0)}(w), a_2^{(0)}(w))$.

We apply the rule for calculating the components of the companion vectors

$$b_i^{(v)} = a_i^{(v)}(D), \quad i = 1, 2; \quad v = 0, 1, \ldots$$

We proceed with the (BGEA) of $a^{(0)}$

$$b^{(0)} = (D + 2D, D^2 + D \cdot D + D^2)$$

$(2^*.3)$ $b^{(0)} = (3D, 3D^2)$.

$$a^{(1)} = (w + 2D - 3D)^{-1} (w^2 + Dw + D^2 - 3D^2, 1) =$$
$$= (w - D)^{-1} (w^2 + Dw - 2D^2, 1) =$$
$$= (w - D)^{-1} ((w - D)(w + 2D), 1).$$

$(2^*.4)$ $a^{(1)} = (w + 2D, w^2 + Dw + D^2) = a^{(0)}$.

By $(2^*.4)$ the (BGEA) of

$$a^{(0)} = ((w + 2D), w^2 + Dw + D^2), \quad w = \sqrt[3]{D^3 + 1}$$

is purely periodic and the length of its primitive period $l = 1$. Using the notation in $(1^*.9)$ we have

$(2^*.5)$ $a_0 = (w + 2D, w^2 + Dw + D^2) = [\overline{3D}, \overline{3D^2}]$

This can be considered the development of the components of a_0 using the periodicity of (BGEA) and those components are algebraic irrationals of third degree. This is not true for all third degree irrationals. But ffor all those third degree

irrationals which are components of a starting vector $a^{(0)}$ that leads to a periodic (BGEA).

3*. Case n = 5.

We denote again

(3*.1) $$w = \sqrt[5]{D^5 + 1}$$

and choose the fixed vector

(3*.2) $$a^{(0)} = (w + 4D,\ w^2 + 3Dw + 6D^2,\ w^3 + 2w^2D + 3D^2w + 4D^3,\ w^4 + Dw^3 + D^2w^2 + Dw^3 + D^4).$$

Using the previous argument we find

(3*.3) $$b^{(0)} = (5D,\ 10D^2,\ 10D^3,\ 5D^4) \text{ or}$$

$$b^{(0)} = (\binom{5}{1}D,\ \binom{5}{2}D^2,\ \binom{5}{3}D^3,\ \binom{5}{4}D^4).$$

Thus the (BGEA) of $a^{(0)}$ is purely periodic with lenghts of primitive period $l = 1$ and like in (2*.5)

(3*.4) $$a_0 = [\overline{5D}, \overline{10D^2}, \overline{10D^3}, \overline{5D^4}\ ].$$

This solves the problem for some fifth degree irrationals.

4*. The general case.

Let w be the irrational

(4*.1) $$w = \sqrt[n]{D^n + 1}\ ;\quad n \geq 2\ ,\quad D \in N\ ;$$

and choose the fixed vector

(4*.2) $$\begin{cases} a^{(0)} = \left(a_1^{(0)}, a_2^{(0)}, ..., a_s^{(0)}, ..., a_{n-1}^{(0)}\right) \\ a_s^{(0)} = \sum_{i=0}^{s} \binom{n - s_i - 1 + i}{i} w^{s-i} D^i \\ s = 1, ..., n-1 \end{cases}$$

The proof that the (BGEA) of the fixed vector $a^{(0)}$ like in (4*.2) is purely periodic and the lenght of its primitive period is $l = 1$ was given by the author before. Thus we find

(4*.3) $$b^{(0)} = (\binom{n}{1}D,\ \binom{n}{2}D^2,\ \ldots,\ \binom{n}{n-1}D^{n-1}).$$

With (4*.3) we denote

(4*.4) $$a_0 = [\overline{nD}, \overline{\frac{n(n-1)}{2}D^2}, ..., \overline{nD^{n-1}}\ ].$$

($4^*.4$) will solve the problem for some n-th degree irrationals w which makes (BGEA) periodic.

This shows that (BGEA) for higher degree irrationals w with $d \mid D$ becomes the General Continued Fraction Algorithm and thus it solves completely Hermite's problem. Since (BGEA) is not periodic for any degree irrational w, we can not get a periodic (BGEA) for any degree irrational and therefore we can not get a General periodic Continued Fraction Algorithm (GCFA) development for every degree irrational w.

(BGEA) gets (BGEA) periodic development for the first time for complex numbers which makes (BGEA) periodic. In conclusion (BGEA) is (GCFA) but it is much more. It extends Hermite's problem over the complex numbers. This problem of Hermite has been open since the middle of the nineteenth century and now (BGEA) solved it. Moreover, it solves its extended version over the complex numbers, where for the first time complex numbers have also a periodic algorithmic development.

Section 3. (BGEA) and the solution of Dirichlet's problem.

Periodicity is a very important property. For instance, in the quadratic case it enables us to solve the so-wrongly called Pellian Equation (it is Euler Equation) $x^2 - my^2 = \pm 1$ or ± 4, where m is a square free natural number, and to find the fundamental unit in the quadratic field $Q(\sqrt{m})$.

Since the group of the fundamental units in the ring of integers in an algebraic number field is isomorphic with the Galois group of a ring with [Jacobson pg. 125] characteristic zero, this problem is called to find the Galois group of units in the algebraic number fields and it is known as Dirichlet's problem.

The problem of finding the multiplicative group of units (the Galois' group) in any algebraic number field A over Q was a difficult open question. For quadratic extensions of Q, the question was completely answered when (ELT) was proved and $x^2 - my^2 = \pm 1$ or ± 4 could be completely solved by simple continued fraction or by (EA). For extensions of Q of degree $n \geq 3$, the problem remained generally unsolved.

Since Dirichlet proved that the group G of units of a number field is finitely generated, many mathematicians have invented ingenious algorithms to calculate fundamental systems of units (the elements of the basis are called fundamental units). A breakthrough in finding explicitly stated units in a wide class of real algebraic function fields took place when Bernstein and Hasse used their algorithm (HBA). They proved that (HBA) applied to some properly chosen vectors in Q(w),

A) $w = \sqrt[n]{D^n + d}$, $d \geq 3$, $d, D \in N$, $d \mid D$, $D \geq (n-2)d$

B) $w = \sqrt[n]{D^n - d}$, $d \geq 3$, $d, D \in N$, $d \mid D$, $D \geq 2(n-1)d$
becomes periodic.
From these results, Hasse and Bernstein proved that in both cases

(3.1) $$e_k = \frac{\overline{w}^k - D^k}{\left(\overline{w} - D\right)^k}, \quad k \mid n, \ k > 1,$$

comprise $\tau(n) - 1$ units in the corresponding fields $Q(w)$. The shortcomings of these very important results are the restriction on d and the bounds on D.

1* All known units from (BGEA).
The basis for this result is

THEOREM 1. Let w be an algebraic integer of degree n. Let

(1*.1) $$a^{(0)} = \left(a_1^{(0)}(w), a_2^{(0)}(w), ..., a_{n-1}^{(0)}(w)\right)$$

$a_i^{(0)}(w)$, $(i = 1, \ldots, n-1)$ be a polynomial in w of degree k, $1 \leq k \leq n-1$, such that $a^{(0)}(w)$ are algebraic integers ;

$a_i^{(0)}(w) \in Q(w, \rho)$; $(i = 1, 2, .., n-1)$.

Let a sequence of vectors of $a^{(v)}$, $(v = 0, 1, \ldots)$ be constructed from $a^{(0)}$ by some (BGEA) such that all companion vectors

(1*.2) $b^{(v)} = (b_1^{(v)}, ..., b_{n-1}^{(v)})$ are integral algebraic vectors $(v = 0, 1, \ldots)$,

$b_i^{(v)} \in Q(w, \rho)$; $(i = 1, 2, .., n-1)$.

Let (BGEA) applied to $a^{(0)}$ become periodic with length of period m so that

(1*.3) $$a^m = a^{(0)}.$$

Then

(1*.4) $$\begin{cases} \prod_{v=0}^{m-1} a_{n-1}^{(v)} \text{ is a unit in the field} \\ Q(w, \tilde{n}), \ \tilde{n} = e^{\frac{2\pi i}{n}}. \end{cases}$$

The field in which the unit $\prod_{v=0}^{m-1} a_{n-1}^{(v)}$ is obtained is the field $Q(w,\rho)$ because carrying out the (BGEA) for $a^{(0)}$ introduces ρ. We get results in $Q(w)$ by choosing $D_1 = D$ to be real.

Proof. The proof is very simple and is based on formulas (1*.5), (1*.6)

(1*.5) $$\prod_{k=1}^{v} a_{n-1}^{(k)} = A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(j+v)} \ ; \quad (v = 1, 2, \ldots);$$

$$(1^*.6)\qquad \begin{vmatrix} 1 & A_0^{(v+1)} & A_0^{(v+2)} & \Lambda & A_0^{(v+n-1)} \\ a_1^{(0)} & A_1^{(v+1)} & A_1^{(v+2)} & \Lambda & A_1^{(v+n-1)} \\ a_2^{(0)} & A_2^{(v+1)} & A_2^{(v+2)} & \Lambda & A_2^{(v+n-1)} \\ M & M & M & & M \\ a_{n-1}^{(0)} & A_{n-1}^{(v+1)} & A_{n-1}^{(v+2)} & \Lambda & A_{n-1}^{(v+n-1)} \end{vmatrix} = \frac{(-1)^{v(n-1)}}{A_0^{(v)} + \sum_{j=1}^{n-1} a_j^{(v)} A_0^{(j+v)}}$$

We first note that, by hypothesis, all $b_i^{(v)}$ $(i = 1, 2, .., n-1 ;\ v = 0, 1, \ldots)$ are algebraic integers. Since the $A_i^{(v)}$ $(i = 1, 2, .., n-1 ;\ v = 0, 1, \ldots)$ are all linear forms in the $b_i^{(v)}$ with integral coefficients, we have

$(1^*.7)$ All $A_i^{(v)}$ $(i = 1, 2, .., n-1 ;\ v = 0, 1, \ldots)$ are algebraic integers and since $a^m = a^{(0)}$ we have

$$(1^*.8)\qquad \begin{cases} a_{n-1}^{(m)} = a_{n-1}^{(0)} \\ \prod_{v=1}^{m} a_{n-1}^{(v)} = \prod_{v=0}^{m-1} a_{n-1}^{(v)} \end{cases}$$

Further we have from $(1^*.5)$

$$(1^*.9)\qquad \prod_{v=1}^{m} a_{n-1}^{(v)} = A_0^{(m)} + \sum_{j=1}^{n-1} a_j^{(m)} A_0^{(j+m)} .$$

Since $a^{(m)} = a^{(0)}$ we have

$$(1^*.10)\qquad a_j^{(m)} = a_j^{(0)},\quad (j = 1, .., n-1) ;$$

From $(1^*.8)$ - $(1^*.10)$ we obtain

$$(1^*.11)\qquad \prod_{v=0}^{m-1} a_{n-1}^{(v)} = A_0^{(m)} + \sum_{j=1}^{n-1} a_j^{(0)} A_0^{(j+m)} .$$

Since by hypothesis of the theorem $A_0^{(j+m)}$ $(j = 0, 1, .., n-1)$ and $a_j^{(0)}$ $(j = 1, .., n-1)$ are algebraic integres, we obtain from $(1^*.11)$

$$(1^*.12)\qquad \begin{cases} e = \prod_{v=0}^{m-1} a_{n-1}^{(v)} \text{ is an algebraic integer} \\ e = A_0^{(m)} + \sum_{j=1}^{n-1} a_j^{(0)} A_0^{(j+m)}. \end{cases}$$

From $(1^*.2)$ we obtain

$$(1^*.13)\qquad e^{-1} = \frac{1}{A_0^{(m)} + \sum_{j=1}^{n-1} a_j^{(m)} A_0^{(j+m)}} = \frac{1}{A_0^{(m)} + \sum_{j=1}^{n-1} a_j^{(0)} A_0^{(j+m)}} =$$

$$= (-1)^{v(n-1)} \begin{vmatrix} 1 & A_0^{(m+1)} & A_0^{(m+2)} & \Lambda & A_0^{(m+n-1)} \\ a_1^{(0)} & A_1^{(m+1)} & A_1^{(m+2)} & \Lambda & A_1^{(m+n-1)} \\ a_2^{(0)} & A_2^{(m+1)} & A_2^{(m+2)} & \Lambda & A_2^{(m+n-1)} \\ M & M & M & & M \\ a_{n-1}^{(0)} & A_{n-1}^{(m+1)} & A_{n-1}^{(m+2)} & \Lambda & A_{n-1}^{(m+n-1)} \end{vmatrix}.$$

But all the entries of the $n \times n$ determinant on the right side of $(1^*.13)$ are algebraic integers, hence the determinant is also an algebraic integer. Since e and e^{-1} are both algebraic integers, e is a unit.

Now all the conditions of the Theorem 2 of Ch. II Section 1 are satisfied here. Therefore, in $Q(w, \rho)$, a unit is given by

$$(1^*.14) \qquad e = \prod_{v=0}^{n(n-1)-1} a_{n-1}^{(v)} \ .$$

Thus, when (BGEA) becomes periodic, an algebraic field unit is generated by the product of the last components of all of the vectors in the cycle.

Hasse and Bernstein first proved this strong result when they got their units from the periodicity of their algorithm (HBA). We proved (making some slight modification) the same result now when we find the units using the periodicity of (BGEA).

Let us give the introductory formulations, and briefly describe the background which leads to our new results in units. We start with

$$(1^*.15) \qquad w = \sqrt[n]{D^n + d}, \quad d \mid D,\ D \in N,\ d \in Z,\ n \geq 3$$

From $(1^*.15)$ we obtain

$$(1^*.16) \qquad w^n - D^n = d$$

hence

$$(1^*.17) \qquad (w - D_1)(w - D_2)\ldots(w - D_n) - d = 0$$

where we use the notation

$$D_k = \rho^k D, \ \rho = e^{\frac{2\delta i}{n}}, \quad k = 1, .., n.$$

From the theory of the n^{th}-roots of unity, ρ^k, $1 \leq k \leq n$, these numbers form a multiplicative group and $\rho\rho^2\ldots\rho^n = 1$. Then the above mentioned D_k, $1 \leq k \leq n$, are all different. Then also different are the linear factors in $(1^*.17)$ with the notations

$$(1^*.18) \qquad f_{i,k}(w) = \prod_{s=i}^{k} (w - D_s)$$

$$f_{i,i}(w) = w - D_i, \quad 1 \leq i \leq k \leq n$$

We then construct the fixed starting vector

$$(1^*.19) \qquad a^{(0)} = (f1, n\text{-}1(w), f1, n\text{-}2(w), .., f_{1,2}(w), f_{2,2}(w))$$

where the first (n-2) components, but not the last component, contain the linear factor $w-D_1$. We rearrange the D_k in the sense that $\{ D_1, D_2, .., D_n \} = \{D, \rho D, \rho^2 D, .., \rho^{n-1} D\}$, so that $D_k = \rho^i D$, k, i = 1, 2, .., n. Let $\{D_1, D_2, .., D_n\}$ be a fixed permutation of $\{D, \rho D, \rho^2 D, .., \rho^{n-1} D\}$.

Then we carry out the (BGEA) of $a^{(0)}$ by the rule

$(1^*.20)$ $\quad b_i^{(v)} = a_i^{(v)}(D_1)$, $i = 1, .., n-1$; $v = 0, 1, \ldots$

Applying $(1^*.20)$ to $(1^*.19)$ we obtain the companion vector since

$a_i^{(0)}(D_1) = 0$, $i = 1, .., n-2$; $a_{n-1}^{(0)}(D) = D_1 - D_2$.

$(1^*.21)$ $\quad b^{(0)} = (0, 0, .., 0, D_1 - D_2)$,

and from here we obtain the vector

$$a^{(1)} = \left(\frac{f_{1,n-2}(w) f_{n,n}(w)}{d}, \frac{f_{1,n-3}(w) f_{n,n}(w)}{d}, \mathrm{K}, \frac{f_{1,1}(w) f_{n,n}(w)}{d}, \frac{f_{n,n}(w)}{d} \right)$$

since $f_{1,n}(w) = d$.

The main point is that in all further steps obtaining $a^{(v)}$, $a_i^{(v)}$,(i=1,..,n -2) contains the linear factor $w - D_1$, but $a_{n-1}^{(v)}$ does not contain that factor, that is, $b_i^{(v)} = 0$, (i=1,..,n -2) and $b_{n-1}^{(v)} = D_1 - D_k$ or $\frac{D_1 - D_k}{d}$, $k \neq 1$, leading finally to periodicity of the algorithm with $a^{(0)} = a^{(n-1)n}$ for $d \neq 1$. The n(n-1) vectors of the primitive period consists of n cycles of vectors each containing n-1 vectors. With the exception of order of d^{-1}, these cycles are all equal. For s = 0, 1, .., n -1, each cycle leads to the product

$$d^{-1} a_{n-1}^{(s(n-1))} a_{n-1}^{(s(n-1)+1)} \ldots a_{n-1}^{(s(n-1)+n-2)},$$

where only one cycle does not have the factor d^{-1}. The components of the $b^{(v)}$ are all zeroes, with the exception of $b_{n-1}^{(v)}$ which has the form $\frac{D_1 - D_k}{d}$ or $D_1 - D_i$, so that the components of the companion vectors are all algebraic integers in view of $d \mid D_m$, m = 1, .., n.

For all this work the polynomial $P(x) = x^n - D^n - d$ was considered for the field equation. Its irreducibility over the field of rationals was proved by the author in Ch. II sec. 1. To obtain Hasse – Bernstein and Halter – Koch and Stender and other units as particular cases of Baica's units from the periodicity of her (BGEA) algorithm we will use another polynomial to be discussed.

We introduce

$$(1^*.22)\qquad \begin{cases} P(x)=\left(\prod_{i=1}^{k}(x^{s_i}-D_i^{s_i})\right)-d; \\ k\geq 2,\ s_i\geq 1;\ D_i\in N;\ d\,|\,D_i; \\ d\in Z;\ i=1,2,...,k;\ |d|\geq 1; \\ 0<D_1<D_2<...<D_k. \end{cases}$$

We shall now prove a series of theorems concerning the polynomial $P(x)$ as defined in $(1^*.22)$.

THEOREM 2. The polynomial $P(x)$ as defined in $(1^*.22)$ is irreducible over the field of rationals in infinitely many cases.

Proof. We first prove

LEMMA 1. Let

$$(1^*.23)\qquad \begin{cases} F(x)=\left(\prod_{i=1}^{k}(x^{s_i}-t_i)\right)-d; \\ k\geq 2,\ s_i\geq 1;\ t_i, d\in Z;\ d\,|\,t_i; \\ i=1,2,...,k;\ |d|\geq 1;\ d \text{ square free.} \end{cases}$$

Then $F(x)$ is irreducible over the field of rationals.

To prove Lemma 1 we first recall Eisenstein's irreducibility criterion. Let $f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$, $a_i\in R$, $i=0,1,..,n$; let p be a prime such that $p\,|\,a_i$, $i=0,1,..,n$; $p^2\,|\,a_n$. Then $f(x)$ is irreducibile over the field of rationals. If we multiply out the factors in $F(x)$ we obtain

$$F(x)=x^n+b_1x^{n-1}+\ldots+b_{n-1}x+b_n$$

$n=s_1+s_2+\ldots+s_k\geq 2$. We have, by condition of $F(x)$,

$d\,|\,b_i$, $i=0,1,..,n$.

$b_n=t_1\cdot t_2\cdot t_3\ldots\cdot t_k-d$.

Now, since $d\,|\,t_i$, $(i=0,1,..,k)$ and $k\geq 2$, we have, since $p\,|\,d$, $p^t\,|\,t_1t_2..t_k$, hence $p^2\,|\,t_1t_2..t_k$. If $p^2\,|\,d_n$, then $p^2\,|\,d$, against presumption. This proved Lemma 1.

Now if we set in $P(x)$; $D_i^{s_i}=t_i$ $(i=0,1,..,k)$, $|d|>1$, d square free, then the conditions of Lemma 1 are fulfilled and thus Theorem 2 is proved for $|d|\neq 1$. For instance, the polynomial $(x-4)(x^3-8^3)(x^5-64^5)-2$ is irreducible, but without

further investigations nothing could be said about the irreducibility of the polynomial $(x^3-15^3)(x^5-40^5)\,(x^8-55^8)-25$ or the polynomial $(x-4)(x^3-8^3)\,(x^5-64^5)-1$. Theorem 2 remains valid if the conditions of $P(x)$ are replaced by $D_i \in Z$, and the magnitude ordering of D_i is dropped. The magnitude ordering of D_I, $0 < D_1 < D_2 < \ldots < D_k$, was introduced in $(1^*.22)$ for convenience.

The reader should also note that without the restriction $|d| > 1$, Lemma 1 could not be applied to prove Theorem 2 .

Now let prove the irreducibility of $P(x)$ for the case $|d| = 1$ also. Since Eisenstein's divisibility criterion does not work in this case, we must appeal to the magnitude criterion of Bernstein. Bernstein proved that :

THEOREM 3. Let the polynomial $F(x)$ be

$$(1^*.24)\qquad \begin{cases} F(x) = x^n + k_1 x^{n-1} + k_2 x^{n-2} + \Lambda + k_{n-1}x - d; \\ d,\ k_j\ (j = 1,\ldots, n-1) \text{ rational integers} \\ d \neq 0;\ \ d \mid k_j;\ |k_{n-1}| \geq c|d|(2+B), \\ B = \sum_{i=0}^{n-2} |k_i|,\ k_0 = 1;\ c \geq 1;. \end{cases}$$

Then $F(x)$ has one, and only one real root, w, which lies in the interval

$$(1^*.25)\qquad \begin{cases} 0 < w < \dfrac{2}{B+4} \ \text{ for } \ \dfrac{k_{n-1}}{d} > 0 \\ \dfrac{-2}{B+4} < w < 0 \ \text{ for } \ \dfrac{k_{n-1}}{d} < 0. \end{cases}$$

The reader will note that all k_j, $j = 1, \ldots, n-2$, can vanish but $|k_{n-1}| \geq 3c|d|$.
Bernstein also proved

THEOREM 4. The polynomial $(1^*.24)$ with $|k_{n-1}| \geq 3|d|B$, $B \geq 2$ is irreducible over Q.

We rearrange the polynomial $(1^*.24)$ as

$$(1^*.26)\qquad \begin{cases} x^n - D^n - d = [(x-D)+D]^n - D^n - d = \\ = \sum_{i=0}^{n-1} \binom{n}{i} D^i (x-D)^{n-i} - d. \end{cases}$$

Our variable from $(1^*.26)$ is now $x - D$ and $k_i = \binom{n}{i} D^i$, $i = 1, \ldots, n-1$.

We stipulate

$$|k_{n-1}| = \binom{n}{n-1} D^{n-1} \geq 2|d| \cdot 2^n \left(1 + D + D^2 + \ldots + D^{n+2}\right)$$

$$= 2^{n+1}|d| \frac{D^{n-1} - 1}{D - 1},$$

and with

$$(1^*.27) \qquad \begin{cases} D \geq 2^{n+1} \\ k_{n+1} = nD^{n-1} \geq \dfrac{2^{n+1}|d|D^{n-1}}{\frac{1}{2}D} \end{cases}$$

$$(1^*.28) \qquad n D \geq 2^{n+2}|d|.$$

Thus we have

THEOREM 5. The number $w = \sqrt[n]{D^n + d}$, $n \geq 2|d|$, $d \mid D$, $D \in N$, $d \in Z$, $D \geq 2^{n+1}$ is an irrational integer of degree n.

Proof. We obtain $n \geq 2|d|$ from $(1^*.27)$ and $(1^*.28)$ stipulating $n D \geq n\, 2^{n+1} \geq 2^{n+2}|d|$. The conditions for B in Theorems 2 and 3 are fulfilled, and thus Theorem 5 is proved. Therefore $x^n - D^n - d$ has a real root and is irreducible over Q for $|d| \geq 1$.

In the above proof the case $|d| = 1$ is included. From $n \geq 2|d|$ it follows that for $|d| = 1$, $n \geq 2$, as it should be. The estimate $D \geq 2^{n+1}$ is an inexact approximation. For example we assume that $\sqrt[n]{2^n + 1}$ is an n-th degree irrational with $D \geq 1$, $|d| = 1$ and $D \geq 2^{n+1}$. The use of Bernstein's irreducibility Theorem is a magnitude criterion, and not a divisibility criterion. Bernstein [I] proved an irreducibility theorem where the condition $d \mid k_I$ $(i = 1, \ldots, n-1)$, k_i from Theorem 3 is dropped.

We now prove the following important result

THEOREM 6. The polynomial P(x) from $(1^*.22)$ has at least one real root $\overline{w}$ if

(1) $d > 0$ $(d \geq 2)$

(2) $d < 0$ $(d \leq -2)$; $k = 1 \pmod 2$

(3) $d < 0$ $(d \leq -2)$; $D_k - D_{k-1} \geq 2|d|$.

Proof.

(1) Here $P(D_1) = - d < 0$;
$P(2D_k) \geq (2D_k - D_1) D_k - d > 0$.
Thus $P(x)$ changes signs at least once between D_1 and $2D_k \neq D_1$.

(2) Here $P(D_1) = - d > 0$.
$P(0) = -d + (-1)^k D_1^{s_1} D_2^{s_2} \cdots D_k^{s_k}$, and
since now $k = 3, 5, \ldots, P(0) \leq - D_1D_2D_3 - d$.
But $D_1D_2D_3 \geq |d|^3$; hence
$P(0) < - |d|^3 - d = - |d|^3 + |d| < 0$.
Thus $P(x)$ changes signs at least once between 0 and $D_1 > 0$.

(3) We have $D_1 \geq |d|$; $D_2 \geq |d|, \ldots, D_{k-1} \geq (k-1)|d|$, $D_k \geq k|d|$.
We have again $P(D_1) = - d > 0$. Let $x = D_k + d$.
Since $D_k - D_{k-1} \geq 2|d|$, $D_i < D_k$, $i = 1, \ldots, k-1$, $D_k - D_i \geq 2|d|$.
Further

$$\prod_{i=1}^{k-1} \left(x^{s_i} - D_i^{s_i}\right) = m \prod_{i=1}^{k-1} (x - D_i),$$

$m \in N$, $m \geq 1$. Thus

$$\prod_{i=1}^{k-1} \left(x^{s_i} - D_i^{s_i}\right) = m \prod_{i=1}^{k-1} (D_k + d - D_i) \geq m \prod_{i=1}^{k-1} (2|d| + d) \geq m|d|.$$

Thus

$$P(x) = \left[\prod_{i=1}^{k} \left(x^{s_i} - D_i^{s_i}\right)\right] - d = \left[\prod_{i=1}^{k-1} \left(x^{s_i} - D_i^{s_i}\right)\right]\left(x^{s_k} - D_k^{s_k}\right) - d =$$

$$= \left[\prod_{i=1}^{k-1} \left(x^{s_i} - D_i^{s_i}\right)\right](x - D_k)t - d, \quad t \in N, \ t \geq 1. \text{ Hence, since } x = D_k + d,$$

$$P(x) = dt \prod_{i=1}^{k-1} \left(x^{s_i} - D_i^{s_i}\right) - d, \text{ and since } d < 0; \ 1 \leq t,$$

$$\prod_{i=1}^{k-1} \left(x^{s_i} - D_i^{s_i}\right) = m|d|, \ m \in N, \ m \geq 1, \ |d| \geq 2,$$

$$P(x) \leq d \, t \, m \, |d| - d < 0.$$

Thus $P(x)$ changes signs between D_1 and $D_k + d \neq D_1$, which proves (3) and thus Theorem 6 is proved.

We did not succeed in eliminating the condition (3) attached to Theorem 6. That this condition (3) is only sufficient is shown by the following example :

$$\text{Let } -d = D_1 = \frac{1}{2} D_2 = D \in N;\ D \geq 2,$$

$$P(x) = (x - D)(x - 2D) + d.$$

This parabola cuts the x-axis in $x_{1,2} = \frac{1}{2}(3D \pm \sqrt{D^2 - 4D})$. Here $D_k - D_{k-1} = 2D - D = D < 2|d|$, but $P(x)$ is still irreducible if D is square free. We have obtained the significant result that $P(x)$, defined by $(1^*.22)$ with the restriction (3) of Theorem 6 is irreducible in infinitely many cases, and has at least one real root in each of these cases. Denoting

$$(1^*.29) \qquad \begin{cases} s_1 + s_2 + \Lambda + s_k = n, \\ s_i \text{ from } (1^*.22) \end{cases}$$

we se that $\overline{w}$, being a root of the irreducible $P(x)$ from $(1^*.22)$ which satisfies condition (3) of Theorem 6 is an n-th degree irrational. If $P(x)$ is formed as stated in $(1^*.22)$, it can have at most n real irrational roots. $P(x)$ cannot have repeated roots nor rational roots in the above case.

We now factor $P(x)$ from $(1^*.22)$ and introduce the notation :

$$(1^*.30) \qquad P(x) = \prod_{j=1}^{k} (x - D_j)(x - \tilde{n}_j D_j) \Lambda\ (x - \tilde{n}_j^{s_j - 1} D_j).$$

$$(1^*.31) \qquad \begin{cases} P(\overline{w}) = 0;\ \tilde{n}_j = e^{\frac{2\delta_i}{s_j}}; \\ \{.., D_j, \tilde{n}_j D_j, \tilde{n}_j^2 D_j, \Lambda, \tilde{n}_j^{s_j - 1} D_j, \Lambda\} \\ = \{\overline{D_j}, \overline{D_2}, \Lambda, \overline{D_n}\}\ j = 1, ..., k. \end{cases}$$

From $(1^*.31)$ we see that any number of the first set of $(1^*.31)$ can equakl any number of its second set. We shall construct an (BGEA) involving the numbers $\overline{D_1}, \overline{D_2}, .., \overline{D_n}$, and once a pairing of these numbers with those of the first set of $(1^*.31)$ has been fixed, we must retain this choice during the process of applying the (BGEA) Algorithm. We shall verify that the number of the two sets in $(1^*.31)$ are all different so these sets are indeed well defined. For let be

$$\rho_u^{\,a} D_u = \rho_v^{\,b} D_v;\ u,v = 1, .., k.$$
$$a = 0, 1, .., s_u - 1;\ \ b = 0, 1, .., s_v - 1.$$

We obtain $|\rho_u^{\,a} D_u| = |\rho_v^{\,b} D_v| \Rightarrow D_u = D_v$, against presumption. Since we shall need the fact that the sets in $(1^*.31)$ are well defined, the ordering of the D_i in $(1^*.22)$ is more plausible. We shall operate with an (BGEA) on the numbers $\overline{w}$,

$\overline{D_1}, \overline{D_2}, .., \overline{D_n}, P(\overline{w}) = 0, \overline{D_i}, (i = 1, .., n)$ from $(1^*.31)$. Let the l.c.m. of $s_1, s_2, .., s_k$ be $\overline{m} = [s_1, s_2, .., s_k]$, $\bar{\eta} = e^{\frac{2\delta i}{m}}$; $Q(\overline{w}, \overline{D_1}, \overline{D_2}, .., \overline{D_n}) = Q(\overline{w}, \bar{\eta})$, where $Q(\overline{w}, D_1) = Q(\overline{w}, D_2) = \ldots = Q(\overline{w}, D_k) = Q(\overline{w})$. We shall now construct a fixed vector $\bar{a}^{(0)} \in Q(\overline{w}, \bar{\eta})$ and then proceed with an (BGEA) in complete analogy with the (BGEA) of $a^{(0)}$.

Let construct the starting vector

$$(1^*.32)\qquad \begin{cases} \bar{a}^{(0)} = \left(f_{1,n-1}(\overline{w}), f_{1,n-2}(\overline{w}), \Lambda, f_{1,2}(\overline{w}), f_{2,2}(\overline{w})\right) \\ f_{i,k}(\overline{w}) = \prod_{s=i}^{k}\left(\overline{w} - \overline{D_s}\right), \\ f_{i,i}(\overline{w}) = \overline{w} - \overline{D_i}, \ 1 \le i \le k \le n \\ P(\overline{w}) = 0; \ P(x) \text{ from } (1^*.22) \\ \text{and irreducible in infinitely many cases.} \end{cases}$$

For the generation of the companion vectors we use the formula

$$(1^*.33)\qquad \overline{b_i}^{(v)} = \overline{a_i}^{(v)}(D_1); \ i - 1, ..., n-1, \ v = 0, 1, ...$$

THEOREM 7. The (BGEA) of $\bar{a}^{(0)}$ from $(1^*.32)$ with the generating formula $(1^*.33)$ for the companion vectors is purely periodic and the length of the primitive period of the (BGEA) equals $m = n(n-1)$ for $d \neq 1$ and $m = n-1$ for $d = 1$.

Apart from slight changes in notation, the proof of Theorem 7 follows those of Theorem 1 and Theorem 2 of Ch.1 sec.1 (4 6) verbatim. We further obtain, as in the case of the (BGEA) for $a^{(0)}$.

COROLLARY 1 to Theorem 7.

The product of the n-1-st components of the n(n-1) vectors of the primitive period of the (BGEA) of $\bar{a}^{(0)}$ equals

$$(1^*.34)\qquad d^{-(n-1)}\left((\overline{w} - D_2)(\overline{w} - D_3)\Lambda\ (\overline{w} - D_n)\right)^n.$$

In the same way we obtain

COROLLARY 2 to Theorem 7.

The components of the n(n-1) companion vectors of (BGEA) of $\bar{a}^{(0)}$ equal

$$(1^*.35)\qquad \begin{cases} \overline{b_i}^{(v)} = 0,\ i = 1,...,n-2;\ v = 0,1,...,n(n-1) \\ \overline{b_{n-1}}^{(v)} = \overline{D_1} - \overline{D_i},\ i = 2,...,n \text{ or} \\ \overline{b_{n-1}}^{(v)} = d^{-1}\left(\overline{D_1} - \overline{D_i}\right),\ i = 2,...,n. \end{cases}$$

Thus all these companion vectors $(1^*.35)$ are algebraic integers.

We now turn to get units in the field $Q(\overline{w}, \overline{\tilde{n}})$ of type K.

We obtain Theorem 8 from Corollaries 1 and 2 to Theorem 7

THEOREM 8. A unit in the field $Q(\overline{w}, \overline{\tilde{n}})$ is given by the expression $(1^*.34)$, viz.,

$$(1^*.36)\qquad \bar{e} = d^{-(n-1)}\left((\overline{w} - \overline{D_2})(\overline{w} - \overline{D_3})K\ (\overline{w} - \overline{D_n})\right)^n.$$

Since $$(\overline{w} - \overline{D_1})(\overline{w} - \overline{D_2})K\ (\overline{w} - \overline{D_n}) = d,$$

$$(1^*.37)\qquad \frac{\left((\overline{w} - \overline{D_1})(\overline{w} - \overline{D_2})K\ (\overline{w} - \overline{D_n})\right)^n}{d^n} = 1$$

so that the expression on the left side of $(1^*.37)$ is equal 1, and is, therefore a unit. Dividing this expression by the expression $(1^*.34)$ we obtain :

$$(1^*.38)\qquad \overline{e_1} = \frac{\left(\overline{w} - \overline{D_1}\right)^n}{d} \quad \text{is a unit in } Q(\overline{w}, \overline{\tilde{n}}).$$

Since $\overline{D_1}$ could be any number out of the n numbers $\overline{D_1}$, $\overline{D_2}$, .., $\overline{D_n}$, we have generally (changing each time the fixed starting vector $\bar{a}^{(0)}$ and the generating rule for the companion vector of the (BGEA) of $\bar{a}^{(0)}$ accordingly):

$$(1^*.39)\qquad \begin{cases} \overline{e_t} = \left(\overline{w} - \overline{D_t}\right)^n,\ t = 1,...,n, \\ \text{are units in } Q(\overline{w}, \overline{\tilde{n}}). \end{cases}$$

For every ρ_i we get units in $Q(\overline{w}, \overline{\tilde{n}_i})$ from (BGEA).

From (BGEA) we obtain more units and now we choose the Si units in $Q(\overline{w}, \overline{\tilde{n}_i})$

$$(1^*.40)\qquad \begin{cases} \overline{\overline{e_{i,0}}} = \dfrac{\left(\overline{w} - D_i\right)^n}{d}, \\ \overline{\overline{e_{i,1}}} = \dfrac{\left(\overline{w} - \tilde{n}_i D_i\right)^n}{d}, \\ \overline{\overline{e_{i,2}}} = \dfrac{\left(\overline{w} - \tilde{n}_i^2 D_i\right)^n}{d}, \mathrm{K}, \\ \overline{\overline{e_{i,S_i-1}}} = \dfrac{\left(\overline{w} - \tilde{n}_i^{S_i-1} D_i\right)^n}{d}. \end{cases}$$

If we multiply all the units of $(1^*.40)$ by each other we obtain the unit

$$(1^*.41)\qquad \overline{\overline{e_i}} = \frac{\left(\left(\overline{w} - D_i\right)\left(\overline{w} - \tilde{n}_i D_i\right)\mathrm{K}\left(\overline{w} - \tilde{n}_i^{S_i-1} D_i\right)\right)^n}{d^{S_i}}$$

We must stress that only the units $(1^*.36)$ were originally obtained by (BGEA); all the others were first obtained by simple algebraic number theoretic considerations from them.

2^* Hasse and Bernstein units from (BGEA).

From the periodicity of (HBA), Hasse and Bernstein [II] prove that

$$(2^*.1)\qquad l_k = \frac{\overline{w}^k - D^k}{\left(\overline{w} - D\right)^k},\ k \mid n,\quad k > 1,$$

are the $\tau(n)-1$ units in the corresponding fields $Q(\overline{w})$. The shortcomings of this very important result at that time are the restriction on d and the bounds on D.

Now we will obtain Hasse and Bernstein units $(2^*.1)$ as particular cases of the author's units from (BGEA) where the restrictions on D are removed.

In $(1^*.40)$ only the units $\overline{\overline{e_{i,0}}}$ where

$$(2^*.2)\qquad \overline{\overline{e_{i,0}}} = \frac{\left(\overline{w} - D_i\right)^n}{d},\quad i = 1, 2, .., k$$

are the Hasse – Bernstein units in $Q(\overline{w})$.

In $(1^*.41)$ we consider

$$(2^*.3)\qquad \overline{\overline{e_i}} = \frac{\left(\overline{w}^{S_i} - D_i^{S_i}\right)^n}{d^{S_i}},\quad (i = 1, 2, .., k) \text{ are also } k \text{ units in } Q(\overline{w}).$$

If we raise $(1^*.40)$ to the power S_i and divide by $(2^*.3)$ we have :

$$(2^*.4)\qquad e_i^{\,*} = \left(\frac{\left(\overline{w} - D_i\right)^{S_i}}{\overline{w}^{S_i} - D_i^{S_i}}\right)^n,\quad (i = 1, 2, .., k)$$

are units in $Q(\overline{w})$.

Therefore, since $\sqrt[n]{e_i^*}$ are units in $Q(\overline{w})$, we obtain that

$$(2^*.5) \qquad \bar{e}_i^* = \frac{\left(\overline{w} - D_i\right)^{S_i}}{\overline{w}^{S_i} - D_i^{S_i}}, \quad (i = 1, 2, \ldots, k)$$

are also units in $Q(\overline{w})$.

For $s_i = 2$ we obtain $\bar{e}_2^* = \dfrac{\overline{w} - D_i}{\overline{w} + D_i}$.

3* Halter – Koch and Stender units from the periodicity of (BGEA).

Halter-Koch and Stender widened the range for d in order to obtain units in $Q(\overline{w})$, but they did not use an algorithm, in proving that the expressions

$$(3^*.1) \qquad e_k = \frac{\overline{w}^k - D^k}{\left(\overline{w} - D\right)^k} \quad \text{are units in } Q(\overline{w}) \text{ if } d \mid D^n.$$

Halter-Koch and Stender units $(3^*.1)$ are also particular cases of Baica's units from (BGEA) where the restrictions on D are removed. Since (BGEA) becomes periodic if $d \mid D$ that will imply that $d \mid D^n$ too, and in $(1^*.40)$ only the units $\overline{e_{i,0}}$ are the Halter-Koch and Stender units as in Hasse and Bernstein units.

4* Halter – Koch units from the periodicity of (BGEA).

Halter-Koch considered the field determined by

$$(4^*.1) \qquad P(x) = \prod_{j=1}^{r_1}\left(x - d_j\right)\prod_{j=r_1+1}^{r_1+r_2}\left(x - z_j\right)\left(x - \bar{z}_j\right) - d$$

with

$$\begin{cases} r_1 \geq 0,\ r_2 \geq 0,\ n = r_1 + 2r_2 \geq 3,\ d, d_j \in Z, \\ d \neq 0;\ d_1 > d_2 > \Lambda > d_{r_1} \end{cases}$$

z_j are integral, complex solutions, $\bar{z}_j$ their conjugates,

$d \mid d_i - d_j$, $d \mid d_i - z_j$, $d \mid z_i - z_j$, $d \mid z_i - \bar{z}_j$ for all possible indices i, j;

if $r_1 = 3$, $r_2 = 0$, $|d| = 2$, then additionally $d_1 - d_2 \geq 4$ or $d_2 - d_3 \geq 4$.

Halter-Koch proved that $P(x)$ has exactly r_1 (different) real zeroes and exactly r_2 (different) pairs of complex conjugate zeroes. To prove this, he needs the additional restrictions

(4*.2) $\left|d_i - d_j\right|,\ \left|d_i - z_j\right|,\ \left|z_i - z_j\right|,\ \left|z_i - \bar{z}_j\right|$ all to be ≥ 2

for all possible indices i, j.

In Q(w), $w = \sqrt[n]{D^n + d}$, $D \in N$, $d \in Z$, $P(w) = 0$, then a complete system of fundamental units consist of $r_1 + r_2 - 1$ elements.

Halter-Koch proves that for $|d| > 1$

(4*.3) $$e_i = \begin{cases} d(w - D_i)^{-n}, & 1 \leq i \leq r_1 \\ d^2\left((w - z_i)(w - \bar{z}_i)\right)^{-n}, & r_1 + 1 \leq i \leq r_1 + r_2 \end{cases}$$

are $r_1 + r_2$ units in Q(w) with $\prod_{j=1}^{r_1+r_2} e_j = 1$, and that any (different) $r_1 + r_2 - 1$ of them form a complete system of independent units in Q(w). For $|d| = 1$, the exponent n in (4*.3) has to be replaced by 1.

In all these results Halter-Koch did not make use of any algorithm.

Let denote

(4*.4) $$\{D_1, D_2, .., D_n\} = \{ d_1, d_2, .., d_{r_1}, z_{r_1+1}, \bar{z}_{r_1+1}, .., z_{r_1+r_2}, \bar{z}_{r_1+r_2} \}$$

where d_i, z_j, $\bar{z}_j$ $(i = 1, .., r_1;\ z_j, \bar{z}_j = r_1 + 1, .., r_1 + r_2)$ are given by (4*.1) and we rewrite P(x) from (4*.1) as follows

(4*.5) $$P(x) = (x - D_1)(x - D_2) \ldots (x - D_n) - d.$$

From here we can apply (BGEA) and we obtain through periodicity of (BGEA) the unit

(4*.6) $$e = \frac{\left(\prod_{i=1}^{n}(w - D_i)\right)^n}{d^{n-1}} = \frac{d}{(w - D_i)^n}.$$

$i = 1, 2, .., r_1, r_1 + 1, .., r_2, r_2 + 1, .., 2r_2$, where $r_1 + 2r_2 = n$.

For $D_i = d_i$ $\quad i = 1, 2, .., r_1$ we obtain $e_i = d(w - d_i)^{-n}$.

For $D_i = z_i$ $\quad i = r_1 + 1, .., r_2$ we obtain $e_i^* = d(w - z_i)^{-n}$ and

for $D_i = \bar{z}_i$ $\quad i = r_1 + 1, .., r_2$ we obtain $\bar{e}_i^* = d(w - \bar{z}_i)^{-n}$. But

$e_i = e_i^* \cdot e_i^* = d^2((w - z_i)(w - \bar{z}_i))^{-n}$ is also a unit and this completes the proof for Halter-Koch units in (4*.3).

We obtained (4*.6) from Theorem 1.

THEOREM 1. A unit in the field $Q(w, \rho)$ is given by the expression :

(4*.7) $$e = d^{-(n-1)}((w - D_2)(w - D_3) \ldots (w - D_n))^n$$

We choose the s_i units in $Q(w, \rho_i)$

$$(4^*.8)\qquad \begin{cases} e_{i,0} = \dfrac{\left(w - D_i\right)^n}{d}, \\ e_{i,1} = \dfrac{\left(w - \tilde{n}_i D_i\right)^n}{d}, \\ e_{i,2} = \dfrac{\left(w - \tilde{n}_i^2 D_i\right)^n}{d}, \\ \qquad \Lambda\ \Lambda\ \Lambda \\ e_{i,s_i-1} = \dfrac{\left(w - \tilde{n}_i^{s_i-1} D_i\right)^n}{d}. \end{cases}$$

and if we multiply all the units in $(4^*.8)$ we obtain the unit

$$(4^*.9)\qquad e_i - \frac{\left(w^{s_i} - D_i^{s_i}\right)^n}{d^{s_i}} \qquad (i = 1, \ldots, k)$$

which are the k units in Q(w). The units $(4^*.7)$ were obtained by (BGEA). The proofs and the results here follow the proofs and the results given before. Therefore, Halter-Koch units are also particular cases of Baica's units from (BGEA).

5* Neubrand units from the periodicity of (BGEA).

Neubrand studied functional or parametric fields of the form

$$(5^*.1)\qquad \{Q(w),\ H(N,w) - bN^l = 0,\ N \in Z\}$$

where

$$(5^*.2)\qquad w = \sqrt[n]{aN^n + bN^l},\quad a = \alpha^n,\ \alpha \in Z,\ b \mid \alpha^n.$$

and

$$(5^*.3)\qquad H(N,w) = \prod_{i=0}^{n-1} h_i(N,w)^{g_i}$$

with

$$(5^*.4)\qquad h_i(N,w) = w^{s_i} + \alpha_{i1} N w^{s_i-1} + K + \alpha_{is_i} N^{s_i}$$

Neubrand proved

THEOREM 1. In functional fields of the form $(5^*.1)$ with

$$(5^*.5)\qquad b^i \mid \alpha_{ji}^n \quad \text{for } j = 0, 1, \ldots, t-1,\ i = 1, \ldots, s_j$$

the elements

$$\varepsilon_{jk} = \frac{h_j(N,w)^{s_k}}{h_k(N,w)^{s_j}} \quad \text{for } j,k = 0, 1, \ldots, t-1$$

are units with the

(5^*.6) $\qquad$ Norm $\varepsilon_{jk} = 1.$

where $h_j(N,w)^{s_k}$, $h_k(N,w)^{s_j}$ are defined as in (5^*.4).

Since
$$\varepsilon_{jk}^{\,n} = \frac{h_j^{ns_k}}{(bN^l)^{s_j s_k}} \cdot \frac{(bN^l)^{s_k s_j}}{h_k^{ns_j}} =$$
$$= \left(\frac{h_j^n}{(bN^l)^{s_j}}\right)^{s_k} \cdot \left(\frac{(bN^l)^{s_k}}{h_k^n}\right)^{s_j}$$

Then

(5^*.7) $\qquad E_j = \dfrac{h_j(N,w)^n}{(bN^l)^{s_j}}$, $\quad j = 0, 1, \ldots, t-1$

is also a unit in $Q(w)$, w as in (5^*.2).

Neubrand's method to derive units is not an algorithmic method. This method is quite different from the other methods and it is algebraic geometric or function theoretic oriented. Using his method later Neubrand got units in quadratic fields.

Nothing is known about the fundamentability of these units and requires further investigations in each case. Stender, who worked in this problem, informed Neubrand that his units ε^* and ε^{**} are also fundamental units.

In (5^*.2) let

(5^*.8) $\qquad bN^l = d$, $\quad (\alpha N)^n = D^n$ and $g_i = 1.$

Then since

(5^*.9) $\qquad H(N,w) = \prod_{i=0}^{n-1} h_i(N,w)^{g_i} = \prod_{i=0}^{n-1} (w^{s_i} - D_i^{s_i})$

(5^*.1) become

(5^*.10) $\qquad \begin{cases} Q(w),\ w = \sqrt[n]{D^n + d},\ P(w) = 0, \\ P(x) \text{ as in } (1^*.22) \end{cases}$

From here, it is obvious that under the conditions of (5^*.8) Neubrand's units can be derived from the periodiciry of (BGEA).

In this section (BGEA) discloses new units of which the already known ones are special cases, and it has the advantage that many results in the theory of units can be derived by means of a unified periodic algorithm.

Finding these new units from the periodicity of (BGEA) the author solved completely Dirichlet's problem.

Dirichlet's problem in higher dimensions $n \geq 3$ do not have solution when (BGEA) fail to be periodic that is when $d \mid D$.

Section 4. (BGEA) and the solution of Galois' theory of polynomials problem.

To find relations between roots and coefficients for higher degree polynomials, as related to Galois' theory of polynomials, was an open problem. The chapters Galois' theory of polynomials and the solvability by radicals are two very important chapters in the algebraic number theory. Proving completely Dirichlet's problem, the Galois' multiplicative group of fundamental units in algebraic number fields give a complete solution to Galois' theory of polynomials, providing the factorization of higher degree polynomials. Once the factorization is known, then we can find the solutions of higher degree polynomial equations. The degree of the equation is related from the solvability by radicals with the dimension of (BGEA) which is equal with the degree of the irrational which makes (BGEA) restrictive periodic. The units in the corresponding algebraic number fields from (BGEA) will give not only the units in the real algebraic number fields but also the units in the complex fields. From the fact that (BGEA) is not always periodic for $n \geq 3$ it follows that not all higher degree polynomial equations can be factorized and solvable by radicals.

Since Galois' theory of polynomials problem is related with the group of units in algebraic number fields $Q(w)$ once that (BGEA) solved Dirichlet's problem then (BGEA) solved the Galois' theory of polynomials problem.

Section 5. (BGEA) and the solution of the Fermat Last Theorem (FLT) problem.

It is well known that $x^2 + y^2 = z^2$ have integral solutions and this is an immediate consequence that (EA) is periodic. It was Hilbert who related for the first time the degree of the above equation with the always periodicity of the (EA). The dimension of the (EA) is $n = 2$ and it is given by the degree of the radical or irrational which makes it periodic. From the solvability by radicals a quadratic equation is solvable by a quadratic irrational in E^2 and every quadratic irrational makes (EA) always periodic (ELT) where the dimension of (EA) is $n = 2$ in the (BGEA) of dimension n.

(5.1) $x^2 + y^2 = z^2$ was solved parametrically for $x = u^2 - v^2$, $y = 2uv$ and $z = u^2 + v^2$, $\gcd(u,v) = 1$ for primitive solutions.

As a consequence that the (EA) is always periodic we have the identity for any u, v

(5.2) $(u^2 - v^2)^2 + (2uv)^2 = (u^2 + v^2)^2$

No such identity as (5.2) for any $n \geq 3$ exists.

The general conjecture is : $x^n + y^n = z^n$ does not have integer solution for $n \geq 3$ and it is known as Fermat Last Theorem (FLT). This conjecture known as (FLT) was stated by Fermat on the margin of his copy of "Euclide's elements" of Bachet's translation of Diophantus, at the side of problem 8 of book 2 "To divide a given square number into two squares". Fermat's marginal note reads:

"To divide a given cube into two cubes, a fourth power, or in general, any power whatever, into two powers of the same denomination above the second is impossible and I have assuredly found an admirable proof of this, but the marginal is too narrow to contain it".

Later, Fermat himself proved his conjecture true for n = 4. He died and no general solution was given, and this problem has baffled the best mathematicians for nearly 360 years.

We will show that the units from (BGEA) play an important role in solving (FLT). The application of units cannot be sufficiently prized. Gauss himself used it to prove the truth of Fermat's conjecture in the cubic case by using units in the quadratic algebraic field $Q(\sqrt{-3})$ and Kummer in his effort to solve (FLT) took refuge in the units of cyclotomic field (complex field as Gauss too). London and Finkelstein have written a whole book yielding information through the theory of units about the famous Mordell equation.

Hasse stated : "The end of the 20-th century will bring the solution for (FLT), and the solution will come from the use of algebraic number theoretical tools, as Fermat had intended."

The original (FLT) was stated in the (EG) and its corresponding number theory is known as the classical algebraic number theory.

In this section we use the restricted periodicity of (BGEA) to prove (FLT) in Euclidean.

Hilbert justified his statement that $x^2 + y^2 = z^2$ have integer solutions because (EA) is always periodic. He related the dimension quadratic with the (EA) and the degree of the equation n = 2 with the solvability by radicals.

(ELT) proves a necessary and sufficient theorem for the always periodicity of the (EA).

Baica and Perron proved an n-dimensional equivalent of (ELT) from quadratics (PBELT). That is a necessary and a sufficient theorem for (BGEA) not always periodic (or restricted periodicity).

It is known in logic, that if

$p \rightarrow q$ then $\sim q \rightarrow \sim p$, but if $p \Leftrightarrow q$ then $\sim p \Leftrightarrow \sim q$.

Hilbert conjecture $x^2 + y^2 = z^2$ have integer solutions if and only if (EA) is always periodic, by generalization becomes that $x^n + y^n = z^n$ have no integer

solutions for $n \geq 3$ because (BGEA) is not always periodic for $n \geq 3$, where $n = 2$ in (BGEA) becomes (EA).

The dimension of (EA) and (BGEA) is given by the degree of the irrational which makes (EA) always periodic and respectively (BGEA) restrictive periodic. Thus, the dimension of the (EA) is 2 and the dimension of (BGEA) is n, and this makes (BGEA) to be the general Euclidean algorithm. With these results we prove the following theorem.

(FLT) THEOREM IN EUCLIDEAN

(BGEA) not always periodic for $n \geq 3$ if and only if Fermat Last Theorem (FLT).

Proof. The proof is done by induction as Fermat originally intended, but at that time he did not have the tool (BGEA) to make the induction on the dimension of (BGEA) and he got stuck after he proved his conjecture true for $n = 4$.

We do not follow the principle of induction step by step since (PBELT) was proved in general, and the principle of induction becomes obvious now.

The (BGEA) is crucial in proving (FLT). I argue that (BGEA) makes the generalization possible. The use of a generalized algorithm and its periodicity were suggested by Hilbert. What I have done is to extend the work of Hasse and Bernstein to obtain tools that can be used to attack Fermat's last theorem (FLT), also. The existence of (BGEA) allows an induction on the degrees of the Fermat's equations. In each case for $n \geq 3$, the (BGEA) is not always periodic. In Euclidean, the principle of induction never fails to give the generalization. The corresponding algebra of the Euclidean geometry (E^nG) is the algebraic number theory (ANT) and that is a Peano-algebra in which induction provides the generalization. Fermat was thinking in the same direction to use induction for $n \geq 3$ but at that time he did not have the tool (BGEA) to be legitimate to use induction in his proof.

Everything that is proved in quadratics from the periodicity of the (EA) are if and only if results.

One immediate consequence of (EA) being always periodic is that $x^2 + y^2 = z^2$ has integral solutions, and this like all of the other results mentioned in the introduction, is an if and only if result. Likewise (BGEA) being the only General Euclidean Algorithm proves up to its restricted periodicity all those similar results in n-dimensions, which like in quadratics, are if and only if results.

Therefore (BGEA) is always periodic if and only if (FLT) is false, and this will bring us to (BGEA) not always periodic for $n \geq 3$ if and only if (FLT) is true. It is an inductive generalization from quadratics and it is the Euclidean proof of (FLT).

In conclusion the proof of (FLT) in Euclidean is the work of Euclid, Jacobi, Perron, Gauss, Euler, Hermite, Hilbert, Dirichlet, Hasse, Bernstein and Baica put together. All of these great mathematicians before me ultimately were looking to

solve the original (FLT) and historically they paved the way for me to finish the final step in its proof. The Euclidean solution of (FLT) is the evolutionary development of the algorithms of Jacobi, Perron, Hasse-Bernstein and Baica.

Section 6. Units from (BGEA) play an important role in solving (FLT)

In this section we will give another justification why Hilbert related the existence of integer solutions for $x^2 + y^2 = z^2$ and the always periodicity of the (EA).

In section 3 we mentioned that the always periodicity of the (EA) is a very important property and in the quadratic case it enables us to solve the Euler-Pell's equations $x^2 - my^2 = \pm 1$ or ± 4, where m is a square free natural number, and to find the fundamental unit in the quadratic field $Q(\sqrt{m})$.

For quadratic extensions of Q, the Galois' group of units was completely answered when the above equations could be completely solved by simple continued fractions or from (EA) always periodicity (ELT). Thus the calculation of units and Dirichlet's problem is related with the solvability of the Diophantine equations.

In this section we will solve explicitly the quadratic Diophantine equation of the form

(6.1) $a^2 \pm ab + b^2 = c^2$

The totality of solutions to (6.1) is given in parameter form by Hasse and they are known as Hasse equations.

No explicit solutions of $a^2 + ab + b^2 = c^2$, $a + b > c > b > a$; $a, b, c \in N \setminus \{0\}$ and $a^2 - ab + b^2 = c^2$, $b > c > a > 0$; $a, b, c \in N \setminus \{0\}$ were known until the author observed that these are homogeneous Diophantine equations and with a proper linear transformation, they can be reduced to a simple Diophantine equation which can be solved explicitly.

This new method of solving $a^2 + ab + b^2 = c^2$ explicitly is to set $a = y - 1$, $b = y + 1$, $y \in N \setminus \{0\}$ and get Euler-Pell's equation $c^2 - 3y^2 = 1$ which can be solved by continued fractions or by the (EA).

To solve $a^2 - ab + b^2 = c^2$, we set $a = \frac{1}{2}(y + 1)$, $b = y - 1$, $y \geq 2$, $y \in N$ and get a corresponding Euler-Pell's equation of the same type $c^2 - 3y^2 = 1$.

The infinite number of solutions in Euler-Pell's equation gives rise to an infinity of solutions to the Hasse's quadratic Diophantine equations.

In the next chapter we will use units derived from (BGEA) to solve more Diophantine equations of higher degree that quadratics.

With this example we give another justification for using the periodicity of (BGEA) in giving solutions of same higher degree Diophantine equations including (FLT).

Section 7. G. Faltings' proof of (FLT) in the geometry of the elliptic curves (GEC).

We know that no "geometry of the elliptic curves" (GEC) and the "arithmetic of the elliptic curves" known as Hecke and Langlands Algebra (HLA) as the number theory of the (GEC) existed 350 years ago, and we are forced to recognize the strong Euclidean character of (FLT), therefore we followed Hasse's advice to solve (FLT) in the algebraic number theory which is the algebra of the Euclidean geometry of dimension n (E^n).

Initially, the 1993 proof of (FLT) in the (GEC) was implemented wrongly in the (ES) of Hecke number theory (HNT) and latter it was abandoned in order to fix the gap.

If the (ES) would not be abandoned in the proof of (FLT) in the (GEC), it could be known as having an equivalent result but not the same with our proof of (FLT) in (E^nG). It is very well known that in order to transfer the same result from one computing algebra system to another computing algebra system there is a need for an analytical continuous transformation (a functor) which is not only a simple transformation needed for an equivalent result. This analytical continuous transformation is known to be the Galois connection needed to transfer a result from one category to the same result in another category that requires analytical continuity.

We will be tempted to recognize Faltings; proof in the (GEC) as an equivalent proof with our Euclidean proof since his commutative diagram with some minor modification will bring us to the solvability by radicals from Euclidean. Therefore, it may be shown that his proof of (FLT) in (GEC) is isomorphic with our proof of (FLT) in (E^nG) which is the original (FLT). It will be the same if the analytical continuous transformation will be provided. Faltings writes [20] "the proof of the conjecture of (FLT) mentioned in the title was finally completed in September of 1994. A. Wiles announced this result in the summer of 1993 ; however, there was a gap in his work. The paper of Taylor and Wiles does not close this gap but circumvents it. This article is an adaptation of several talks that I have given on this topic and is by no means about my own work. I have tried to present the basic ideas to a wider mathematical audience, and in the process I have skipped over certain details, which are in my opinion not so much of interest to the non-specialists. The specialists can then alleviate their boredom by finding those mistakes and correcting them."