CHAPTER IV - MORE IMPORTANT APPLICATIONS OF (BGEA)

Section 0. Introduction.

In this chapter we will use units derived from (BGEA) to solve diophantine equations unsolved before, we will approximate irrationals by the periodic (BGEA) algorithm and obtain n-dimensional Fibonacci numbers.

Section I. (BGEA) Algorithmic approximation of some second, third and fifth degree irrationals

This section deals with approximation of irrationals of degree n=2,3,5. Most simple rational fractions are approximation irrationals, especially of the type $\sqrt{D^2+1}$, $\sqrt[3]{D^3+1}$ and $\sqrt[5]{D^5+1}$, with d=1, leading to the most simple ones of $\sqrt{2}, \sqrt[3]{2}, \sqrt[5]{2}$. Through approximations of these irrationals in a variety of patterns are known, the results under discussion here are new and practical.
The main algebraic machine-tool which is the starting point and the main ingredient of this result is again (BGEA) algorithm.

DEFINITION 1. We form the numbers

(1.1) $$A_i^{(v)} = \delta_i^v, \text{ the Kronecker delta } i,v=0,1,\ldots n-1$$

Let a (BGEA) of $a^{(0)}$ hold. The recursion formula

(1.2) $$A_i^{(n+v)} = \sum_{j=0}^{n-1} b_j^v A_i^{(v+j)}, b_0^{(v)} = 1; v = 0,1...$$

generates the "matricians" of the (BGEA) of $a^{(0)}$. L.Bernstein [I] has proved the following formulas for the real algebraic fields, and the author [1] proved that they are also true for the complex fields.

(1.3) $$\begin{vmatrix} A_0^{(v)}, A_0^{(v+1)}, ..., A_0^{(v+n-1)} \\ A_1^{(v)}, A_1^{(v+1)}, ..., A_1^{(v+n-1)} \\ ... \quad ... \quad ... \quad ... \\ A_{n-1}^{(v)}, A_{n-1}^{(v+1)}, ..., A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)}$$

(1.4) $$a_i^{(0)} = \frac{\sum_{j=1}^{n-1} a_j^{(v)} A_i^{(v+j)}}{\sum_{j=0}^{n-1} a_j^{(v)} A_0^{(v+j)}}; a_0^{(v)} = 1, \left.\begin{matrix} i = 0,1,...; \\ v = 0,1,... \end{matrix}\right\}$$

Perron [VIII] proved the following theorem which is a generalization of the original theorem on convergence of simple continued fractions. In our terminology this would yield.

THEOREM 1. Let a (BGEA) of some $a^{(0)} \in E_{n-1}$ hold. If the components of all companion vectors are positive and the (BGEA) of this $a^{(0)}$ is periodic then

(1.5) $$a_i^{(0)} = \lim_{v\to\infty} \frac{A_i^{(v)}}{A_0^{(v)}}, i = 0,1,...$$

where $A_i^{(v)}$, (i=0,1,…n-1; v=0,1,…) are the matricians of the (BGEA) of $a^{(0)}$.

The fractions $\frac{A_i^{(v)}}{A_0^{(v)}}$ are called the "convergents" of the (BGEA) of $a^{(o)}$.

The reader should note that if the (BGEA) of some $a^{(0)}$ is periodic then exists an $a^{(v)}$ in this (BGEA) such that the (BGEA) of $a^{(v)}$ is purely periodic. With this in mind the author [1] has proved.

THEOREM 2. Let

(1.6) w an n-th degree integer (n ≥ 2),

and $a^{(0)}$ a fixed vector such that

(1.7) $$\left.\begin{matrix} a^{(0)} = (a_1^{(0)}(w), a_2^{(0)}(w), ..., a_{n-1}^{(0)}(w)) \\ a_i^{(0)}(w) \text{ algebraic integers } (i = 1,...n-1). \end{matrix}\right\}$$

Let the (BGEA) of $a^{(0)}$ be purely periodic with length of the primitive period l. Let the components of the companion vectors be algebraic integers. Then

(1.8) $$\left.\begin{matrix} A_0^{(vl)} + a_1^{(0)} A_0^{(vl+1)} + a_2^{(0)} A_0^{(vl+2)} + ... + a_{n-1}^{(0)} A_0^{(vl+n-1)} \\ v = 1,2,.... \end{matrix}\right\}$$

and units, namely the v-th powers of

$$A_0^{(l)} + a_1^{(0)} A_0^{(l+1)} + a_2^{(0)} A_0^{(l+2)} + ... + a_{n-1}^{(0)} A_0^{(l+n-1)}.$$

In this context we shall need the formula which was also used the author before in Chapter 2.

(1.9) $$\prod_{i=1}^{k} a_{n-1}^{(i)} = A_0^{(k)} + a_1^{(k)} A_0^{(k+1)} + \ldots + a_{n-1}^{(k)} A_0^{(k+n-1)}$$

If the (BGEA) of $a^{(0)}$ in E_{n-1} is purely periodic with length of the primitive period l, then it follows from (1.9)

(1.10) $$e = \prod_{i=0}^{l-1} a_{n-1}^{(i)} = \sum_{j=0}^{n-1} a_1^{(0)} A_0^{(l+j)} \quad , \text{ e a unit}$$

since in this case $a_{n-1}^{(l)} = a_{n-1}^{(0)}$.

We also have the formula, in virtue of (1.10)

(1.11) $$\left. \begin{array}{c} e^f = A_0^{(fl)} + a_1^{(0)} A_0^{(fl+1)} + \ldots + a_{n-1}^{(0)} A_0^{(fl+n-1)} \\ f = 1,2,\ldots \end{array} \right\}$$

1*.**Approximation of irrationals - case n=2**

Through this case is well known from the expansion of real quadratic irrationals as simple continued fractions, we shall include it in our discussion.

Let

(1*.1) $$w = \sqrt{D^2 + 1}, D \in N, \text{ w a quadric irrational}$$

That w is irrational (for D>0) is banal.

We chose the fixed vector

(1*.2) $$a^{(0)} = w + D,$$

since here n-1=1. Thus $a_1^{(0)} = a_{n-1}^{(0)}$, and we shall generally denote

(1*.3) $$a^{(v)} = a_v, v = 0,1,\ldots; a_v = a_v(w) \text{ for all (BGEA) of } a^{(0)}.$$

In conformity with (1*3) we shall also denote

$$b^{(v)} = b_v, \; v = 0,1\ldots$$

For the calculation of the companion vectors we use the rule

(1*.4) $$b^{(v)} = b_v = a_v(D), \; v = 0,1,\ldots$$

and have

(1*.5) $$b_0 = (w + D)_{w=D} = 2D$$

hence, by (1*.1)

(1*.6) $$a_1 = [(w + D) - 2D]^{-1} \cdot 1 = (w - D)^{-1} = w + D$$

since $(w - D)^{-1} = (w + D)$ from $w^2 - D^2 = 1$. Thus

(1*.7) $$a_0 = a_1 \ldots = a_v, \; v = 0,1,\ldots$$

and the (BGEA) of $a_0 = w + D$ is purely periodic with length of the primitive period *l*=1. Further, in this case,

(1*.8) $$[a_v] = [w + D] = [w] + D = 2D = b_v$$

the (BGEA) of w+D coincides with the Euclidian algorithm, and we have, in the notation of continued fractions

$$a_0 = w + D = [2\overline{D}]$$

For the calculation of the matricians of $a^{(0)}$ we have from (1.1) and (1.2)

(1*.9) $$\left.\begin{array}{l} A_0^{(0)} = 1, A_0^{(1)} = 0, A_0^{(n+2)} = A_0^{(n+2)} = A_0^{(n)} + 2DA_0^{(n+1)}, \\ A_1^{(0)} = 0, A_1^{(1)} = 1, A_1^{(n+2)} = A_1^{(n)} + 2DA_1^{(n+1)}, \end{array}\right\}$$

$n = 0,1,...$

Formula (1.4) yields

(1*.10) $$w + D = \frac{A_1^{(v)} + (w + D)A_1^{(v+1)}}{A_0^{(v)} + (w + D)A_0^{(v+1)}},$$

(1*.11) $$(w + D)A_0^{(v)} + (w + D)^2 A_0^{(v+1)} = A_1^{(v)} + (w + D)A_1^{(v+1)},$$

and comparing in (1*.10) coefficients of w (namely the highest irrational power of w), we obtain

(1*.12) $$A_1^{(v+1)} = A_0^{(v)} + 2DA_0^{(v+1)} = A_1^{(v+2)}$$

2*. **Explicit representation of the matricians**

We shall give an explicit representation of $A_0^{(v)}$ $(v = 2, 3,...)$.

By formula (1*12) which, because of (1*11) will also provide an explicit representation of $A_1^{(v)}$ $(v = 2,3,...)$, we obtain from (1*.8), by means of Euler's function

$$\sum_{i=0}^{\infty} A_0^{(i)} x^i = A_0^{(0)} + A_0^{(1)} x + \sum_{i=2}^{\infty} A_0^{(i)} x^i = 1 + \sum_{i=0}^{\infty} A_0^{(i+2)} x^{i+2} =$$

$$= 1 + \sum_{i=0}^{\infty} (A_0^{(i)} + 2DA_0^{(i+1)}) x^{i+2} = 1 + x^2 \sum_{i=0}^{\infty} A_0^{(i)} x^i + 2Dx \sum_{i=0}^{\infty} A_0^{(i+1)} x^{i+1} =$$

$$= 1 + x^2 \sum_{i=0}^{\infty} A_0^{(i)} x^i + 2Dx\,(-A_0^{(0)} + \sum_{i=0}^{\infty} A_0^{(i)} x^i).$$

Hence

$$(1 - x^2 - 2Dx)\sum_{i=0}^{\infty} A_0^{(i)} x^i = 1 - 2Dx,$$

$$\sum_{i=0}^{\infty} A_0^{(i)} x^i = \frac{1-2Dx}{1-(x^2+2Dx)} = 1 + \frac{x^2}{1-(x^2+2Dx)},$$

$$A_0^{(0)} + A_0^{(1)} x + \sum_{i=2}^{\infty} A_0^{(i)} x^i = 1 + \sum_{i=0}^{\infty} A_0^{(i+2)} x^{i+2} = \frac{x^2}{1-(x^2+2Dx)},$$

$$\sum_{i=0}^{\infty} A_0^{(i+1)} x^i = \sum_{k=0}^{\infty} (x^2+2Dx)^k.$$

(x sufficiently small)

Choosing i=n, we obtain by comparison of coefficients of x^n

$$A_0^{(n+2)} = \sum \binom{y_1 + y_2}{y_1, y_2} (2D)^{y_2} x^{2y_1 + y_2}$$

But $2y_1+y_2 = n$, $y_1+y_2 = n-y_1$,

$$A_0^{(n+2)} = \sum \binom{n - y_1}{y_1} (2D)^{n-2y_1}$$

(2*.1)
$$\left. \begin{array}{c} A_0^{(n+2)} = \sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n-i}{i} (2D)^{n-2i} \\ n = 0,1... \end{array} \right\}$$

We obtain from (2*.1), for $n \Rightarrow 2n-2$

(2*.2)
$$A_0^{(2n)} = \sum_{i=0}^{n-1} \binom{2n-2-i}{i} (2D)^{2n-2-2i}, \quad n = 1,2,...$$

and for $n \Rightarrow 2n-1$

(2*.3)
$$\left. \begin{array}{c} A_0^{(2n+1)} = \sum_{i=0}^{n-1} \binom{2n-1-i}{i} (2D)^{2n-1-2i} \\ n = 1,2,... \end{array} \right\}$$

We shall verify formulas (2*.2), (2*.3), comparing the results with these from (1*.8).

We obtain from the latter

$$A_0^{(2)} = 1, A_0^{(3)} = A_0^{(1)} + 2DA_0^{(2)} = 2D,$$
$$A_0^{(4)} = A_0^{(2)} + 2DA_0^{(3)} = 1 + 4D^2;$$
$$A_0^{(5)} = A_0^{(3)} + 2DA_0^{(4)} = 4D + 8D^3;$$
$$A_0^{(6)} = A_0^{(4)} + 2DA_0^{(5)} = 1 + 12D^2 + 16D^4.$$

From (2*.2) we obtain for n=1,2,3

(2*.4) $$A_0^{(2)} = 1, A_0^{(4)} = 1 + 4D^2; A_0^{(6)} = 1 + 12D^2 + 16D^4.$$

From (2*.3) we obtain for n=1,2,3

(2*.5) $$A_0^{(3)} = \sum_{i=0}^{n-1} \binom{2n-1-i}{i} (2D)^{(2n-1-2i)} = 2D$$

$$A_0^{(5)} = \sum_{i=0}^{1} \binom{2n-1-i}{i} (2D)^{(2n-1-2i)} = \binom{2}{1} 2D + (2D)^3$$

$$A_0^{(5)} = 4D + 8D^3.$$

$$A_0^{(7)} = \sum_{i=0}^{2} \binom{5-i}{i} (2D)^{5-2i} = \binom{3}{2} 2D + \binom{4}{1} (2D)^3 + (2D)^5,$$

$$A_0^{(7)} = 6D + 32D^3 + 32D^5.$$

From (2*.3) we obtain

(2*.6) $$A_0^{(7)} = A_0^{(5)} + 2DA_0^{(6)} = 4D + 8D^3 + 2D\,(1 + 12D^2 + 16D^4)$$
$$= 6D + 32D^3 + 32D^5.$$

3*. **The convergents of $\sqrt{2}$**

We obtain from formula (1.5), since in our case n=2,

$$a^{(0)} = a_1^{(0)} = \lim_{v \to \infty} \frac{A_1^{(v)}}{A_0^{(v)}},$$

(3*.1) $$w + D \approx \frac{A_1^{(m+1)}}{A_0^{(m+1)}}, \; m = 1,2...$$

With (1*.11), we obtain from (3*.1)

(3*.2) $$w + D \approx \frac{A_0^{(m)} + 2DA_0^{(m+1)}}{A_0^{(m+1)}} = 2D + \frac{A_0^{(m)}}{A_0^{(m+1)}}$$

(3*.3) $$w \approx D + \frac{A_0^{(m)}}{A_0^{(m+1)}}.$$

We investigate the special case D=1 and obtain from (3*.3)

$$m = 2, \sqrt{2} \approx 1 + \frac{1}{2} = 1.5$$

$$m = 3, \sqrt{2} \approx 1 + \frac{2}{5} = 1.4$$

$$m = 4, \sqrt{2} \approx 1 + \frac{5}{12} = 1.416$$

$$m = 5, \sqrt{2} \approx 1 + \frac{12}{29} = 1.413$$

$$m = 6, \sqrt{2} \approx 1 + \frac{29}{70} = 1.414$$

Thus for m=5,6 we already obtain quite a tolerable approximation for $\sqrt{2}$.

As known, $\sqrt{5}$ occupies an exceptional place in number theory. We set D=2, and obtain, with formula (3*.3)

(3*.4) $$w = \sqrt{5} \approx 2 + \frac{A_0^{(m)}}{A_0^{(m+1)}}.$$

We obtain from previous calculations of the matricians with D=2:

$$A_0^{(3)} = 4, A_0^{(4)} = 17, A_0^{(5)} = 72, A_0^{(6)} = 305, A_0^{(7)} = 1294$$

we obtain the approximation values

$$\sqrt{5} \approx 2 + \frac{4}{17}, \approx 2 + \frac{17}{72}, \approx 2 + \frac{72}{305}, \approx 2 + \frac{305}{1294}$$

$$\sqrt{5} \approx 2.235, \approx 2.235, \approx 2.235.$$

Thus 2.236 is a good approximation of $\sqrt{5}$. That $\left(A_0^{(m)}, A_0^{(m+1)}\right) = 1$ follows from (1.3)

4*. **The case n=3**

We denote again

(4*.1) $$w = \sqrt[3]{D^3 + 1}$$

and choose the fixed vector

(4*.2) $$a^{(0)} = (w + 2D, w^2 + Dw + D^2).$$

With $a^{(0)} = (a_1^{(0)}(w), a_2^{(0)}(w))$, we again apply the rule for calculating the components of the companion vectors

$$b_i^{(v)} = a_i^{(v)}(D), i = 1,2; v = 0,1,...$$

We proceed with the (BGEA) of $a^{(0)}$.

$$b^{(0)} = (D + 2D, D^2 + D.D + D^2)$$

(4*.3)

$$b^{(0)} = (3D, 3D^2)$$

$$a^{(1)} = (w + 2D - 3D)^{-1}(w^2 + Dw + D^2 - 3D^2, 1)$$

$$= (w - D)^{-1}(w^2 + Dw - 2D^2, 1) =$$

$$= (w - D)^{-1}((w - D)(w + 2D), 1),$$

(4*.3) $$a^{(1)} = (w + 2D, w^2 + Dw + D^2) = a^{(0)}.$$

By (4*.4) the (BGEA) of $a^{(0)} = ((w + 2), w^2 + wD + D^2), w = \sqrt[3]{D^3 + 1}$ is purely periodic and the length of its primitive period *l*=1. We shall proceed to calculate the matricians of the (BGEA) of this $a^{(0)}$.

We have with $b^{(0)} = b^{(v)} = (3D, 3D^2), v = 1,2,...$

(4*.5) $$\left.\begin{array}{c} A_0^{(0)} = 1, A_0^{(1)} = A_0^{(2)} = 0, A_0^{(n+3)} = A_0^{(n)} + 3DA_0^{(n+1)} + 3D^2A_0^{(n+2)} \\ n = 0,1,... \end{array}\right\}$$

$$\sum_{n=0}^{\infty} A_0^{(n)}x^n = A_0^{(0)} + A_0^{(1)}x + A_0^{(2)}x^2 + \sum_{n=0}^{\infty} A_0^{(n+3)}x^{n+3}$$

$$= 1 + x^3\sum_{n=0}^{\infty} A_0^{(n+3)}x^n =$$

$$= 1 + x^3\sum_{n=0}^{\infty}(A_0^{(n)} + 3DA_0^{(n+1)} + 3D^2A_0^{(n+2)})x^n =$$

$$= 1 + x^3\sum_{n=0}^{\infty} A_0^{(n)}x^n + 3Dx^2\sum_{n=0}^{\infty} A_0^{(n+1)}x^{n+1} + 3D^2x\sum_{n=0}^{\infty} A_0^{(n+2)}x^{n+2} =$$

$$= 1 + x^3\sum_{n=0}^{\infty} A_0^{(n)}x^n + 3Dx^2\left(-A_0^{(0)} + \sum_{n=0}^{\infty} A_0^{(n)}x^n\right) +$$

$$+ 3D^2x\left(-A_0^{(0)} - A_0^{(1)}x + \sum_{n=0}^{\infty} A_0^{(n)}x^n\right) =$$

$$= 1 + (x^3 + 3Dx^2 + 3D^2x)\sum_{n=0}^{\infty} A_0^{(n)}x^n - 3Dx^2 - 3D^2x,$$

$$\sum_{n=0}^{\infty}[1-(3D^2x+3Dx^2+x^3)]A_0^{(n)}x^n=1-3D^2x-3Dx^2,$$

$$\sum_{n=0}^{\infty}A_0^{(n)}x^n=\frac{1-(3D^2x+3Dx^2)}{1-(3D^2x+3Dx^2+x^3)},$$

$$\sum_{n=0}^{\infty}A_0^{(n)}x^n=1+\frac{x^3}{1-(x^3+3Dx^2+3D^2x)},$$

and as before

(4*.6) $$\sum_{n=0}^{\infty}A_0^{(n+3)}x^n=\sum_{k=o}^{\infty}(x^3+3Dx^2+3D^2x)^k$$

for sufficiently small x.

Comparing coefficients of x^n on both sides (4^*.6) we obtain

$$A_0^{(n+3)}=\sum_{3y_1+2y_2+y_3=n}\binom{y_1+y_2+y_3}{y_1,\ y_2,\ y_3}(3D)^{y_2}(3D^2)^{y_3}$$

(4*.7) $$\left.\begin{array}{l} A^{(n+3)}=\displaystyle\sum_{3y_1+2y_2+y_3=n}\binom{y_1+y_2+y_3}{y_1,\ y_2,\ y_3}3^{y_2+y_3}3D^{y_2+2y_3} \\ n=0,1,...;\binom{0}{0}\equiv 1. \end{array}\right\}$$

From (4*.7) the matricians are easily calculated, finding y_1, y_2, y_3, from the simple linear equations $3y_1+2y_2+y_3=n$. One proceeds in a lexiconographic order. We have

$$n=0;\ y_1=y_2=y_3=0;\ A_0^{(3)}=1;$$

$$n=1;\ y_3=1,\ y_1=y_2=0;\ A_0^{(4)}=3D^2;$$

$$n=2;\ y_1=0=y_3,\ y_2=1;\ y_1=y_2=0,\ y_3=2;$$

$$A_0^{(5)}=3D+9D^4;$$

$$n=3;\ y_1=1,\ y_2=0,\ y_3=0;\ y_1=0,\ y_2=1,\ y_3=1;$$

$$y_1=y_2=0,\ y_3=3;$$

$$A_0^{(6)}=1+2\cdot 3^2D^3+3^3D^6,$$

$$A_0^{(6)}=1+18D^3+27D^6.$$

$$n=4;\ y_1=1,\ y_2=0,\ y_3=1;\ y_1=0,\ y_2=2,\ y_3=0;$$

$$y_1=0,\ y_2=1,\ y_3=2;\ y_1=y_2=0,\ y_3=4;$$

$$A_0^{(7)} = 2\cdot 3D^2 + 1\cdot 3^2 D^2 + \binom{3}{2}\cdot 3D(3D^2)^2 + 3^4 D^8$$

$$A_0^{(7)} = 15D^2 + 81D^5 + 81D^8.$$

One could verify these results by means of formula (4*.5). From this we obtain

$$A_0^{(8)} = A_0^{(5)} + 3DA_0^{(6)} + 3DA_0^{(7)} = 3D + 9D^4 +$$
$$+ (1 + 18D^3 + 27D^6)3D + 3D^2(15D^2 + 81D^5 + 81D^8) =$$
$$= 6D + 108D^4 + 324D^7 + 243D^{10}.$$
$$A_0^{(9)} = A_0^{(6)} + 3DA_0^{(7)} + 3D^2 A_0^{(8)} =$$
$$= 1 + 18D^3 + 27D^6 + 3D(15D^2 + 81D^5 + 81D^8) +$$
$$+ 3D^2(6D + 108D^4 + 324D^7 + 243D^{10}).$$
$$A_0^{(9)} = 1 + 81D^3 + 594D^6 + 1215D^9 + 729D^{12}.$$

It would be an interesting problem whether calculating preference should be given to formula (4*.5) or (4*.7)

We shall shortly discuss the number of solutions of $3y_1+2y_2+y_3=3n$

$y_1 = n;\ y_2 = y_3 = 0$ — 1 solution

$y_1 = n-1;\ y_2 = 1, y_3 = 1;\ y_2 = 0, y_3 = 3;$ — 2 solutions

$y_1 = n-2;\ y_2 = 3, y_3 = 0;\ y_2 = 2, y_3 = 2;$
$y_2 = 1, y_3 = 4;\ y_2 = 0, y_3 = 6;$ — 4 solutions

$y_1 = n-3;\ y_2 = 4, y_3 = 1;\ y_2 = 3, y_3 = 3;$
$y_2 = 2, y_3 = 5;\ y_2 = 1, y_3 = 7;$ — 5 solutions
$y_2 = 0, y_3 = 9;$

Thus the number of solutions of $3y_1 + 2y_2 + y_3 = 3n$ equals

$$S_{3n} = 1 + (2+4) + (5+7) + (8+10) + (11+13) + \ldots \quad \text{(n numbers)}$$
$$S_{3n} = (1+4+7+\ldots) + (2+5+8+11+\ldots)$$
$$\approx \left(\left[2 + \left(\left[\frac{n}{2}\right] - 1\right)3\right] + \left[2.2 + \left(\left[\frac{n}{2}\right] - 1\right)3\right]\right)\left[\frac{n}{2}\right] \approx \left[\frac{3n^2}{2}\right]$$

Thus, approximately

(4*.8) $$S_{3n} \approx \left[\frac{3n^2}{2}\right]$$

For n=2, $S_6 = 7 \approx \frac{3 \cdot 2^2}{2} = 6$

From the other side, $A_0^{(3n)}$ contains exactly 2(n-1)+1=2n-1 summands, as the reader can easily verify.

In order to calculate $A_0^{(3n)}$ by formula (4*.5), one has to calculate the preceding 3n matricians $A_0^{(0)}, A_0^{(1)}, \ldots, A_0^{(3n-1)}$, so the number of these manipulations equals 3n(2n-1). The author therefore conjectures, that it is preferable to calculate the matricians, from a time and operation saving profit view, by formula (4*.7).

We shall now calculate the matricians $A_1^{(v)}$ and $A_2^{(v)}$, expressing them as linear functions of $A_0^{(t)}$. We have, with $a^{(1)} = a^{(0)} = (w + 2D, w^2 + Dw + D^2)$, and in virtue of formula (1.4)

$$w + 2D = \frac{A_1^{(n)} + (w + 2D)A_1^{(n+1)} + (w^2 + Dw + D^2)A_1^{(n+2)}}{A_0^{(n)} + (w + 2D)A_0^{(n+1)} + (w^2 + Dw + D^2)A_0^{(n+2)}}$$

(4*.9)
$$\left.\begin{aligned} (w + 2D)A_0^{(n)} + (w + D)A_0^{(n+1)} + (w^2 + Dw + D^2)A_0^{(n+2)} = \\ A_1^{(n)} + (w + D)A_1^{(n+1)} + (w^2 + Dw + D^2)A_1^{(n+2)}. \end{aligned}\right\}$$

Comparing the coefficients of the powers of the irrational w^2 on both sides of (4*.9), we obtain

(4*.10)
$$A_1^{(n+2)} = A_0^{(n+1)} + 3DA_0^{(n+2)}$$

In the same way we obtain from

$$(w^2 + Dw + D^2)A_0^{(n)} + (w + 2D)A_0^{(n+1)} + (w^2 + Dw + D^2)A_0^{(n+2)} =$$
$$= A_2^{(n)} + (w + 2D)A_2^{(n+1)} + (w^2 + Dw + D^2)A_2^{(n+2)}$$

(4*.11)
$$A_2^{(n+2)} = A_0^{(n)} + 3DA_0^{(n+1)} + 3D^2A_0^{(n+2)}$$

5*. The convergents of $\sqrt[3]{D^3 + 1}$

We obtain in case n=3 with $w = \sqrt[3]{D^3 + 1}$,

$$a_1^{(0)} = w + 2D, a_2^{(0)} = w^2 + Dw + D^2,$$

and by formulas (1.5), (4*.10), (4*.11)

$$w + 2D = \lim_{n\to\infty} \frac{A_1^{(n+1)}}{A_0^{(n+1)}} = \lim_{n\to\infty} \frac{A_0^{(n)} + 3DA_0^{(n+1)}}{A_0^{(n+1)}} = 3D + \lim_{n\to\infty} \frac{A_0^{(n)}}{A_0^{(n+1)}}.$$

(5*.1)
$$w = D + \lim_{n\to\infty} \frac{A_0^{(n)}}{A_0^{(n+1)}}.$$

Substituting in (5*.1) the values for the matricians $A_0^{(v)}$ from (4*.7), we obtain with $n \Rightarrow n+3$.

(5*.2)
$$w = D + \lim_{n\to\infty} \frac{\sum\limits_{3y_1+2y_2+y_3=n} \binom{y_1+y_2+y_3}{y_1, y_2, y_3} 3^{y_2+y_3} D^{y_2+2y_3}}{\sum\limits_{3y_1+2y_2+y_3=n+1} \binom{y_1+y_2+y_3}{y_1, y_2, y_3} 3^{y_2+y_3} D^{y_2+2y_3}}$$

(5*.2) is a very interesting and simple formula for calculating the convergents of $\sqrt[3]{D^3+1}$.

We further obtain by the same method

$$w^2 + Dw + D^2 = \lim_{n\to\infty} \frac{A_2^{(n+2)}}{A_0^{(n+2)}} = \lim_{n\to\infty} \frac{A_0^{(n)} + 3DA_0^{(n+1)} + 3D^2 A_0^{(n+2)}}{A_0^{(n+2)}} =$$

$$= 3D^2 + \lim_{n\to\infty} \frac{A_0^{(n)}}{A_0^{(n+2)}} + 3D \lim_{n\to\infty} \frac{A_0^{(n+1)}}{A_0^{(n+2)}} = w^2 + D(D + \lim_{n\to\infty} \frac{A_0^{(n+1)}}{A_0^{(n+2)}}) =$$

$$= w^2 + D(D + \lim_{n\to\infty} \frac{A_0^{(n+1)}}{A_0^{(n+2)}}) = 2D^2 + \lim_{n\to\infty} \frac{A_0^{(n)}}{A_0^{(n+2)}} + 3D \lim_{n\to\infty} \frac{A_0^{(n+1)}}{A_0^{(n+2)}},$$

$$(5^*.3)\quad \left.\begin{aligned} w^2 &= D^2 + \lim_{n\to\infty}\frac{A_0^{(n)}}{A_0^{(n+2)}} + 2D\lim_{n\to\infty}\frac{A_0^{(n+1)}}{A_0^{(n+2)}} \\ w^2 &= D^2 + \lim_{n\to\infty}\left|\frac{\sum\limits_{3y_1+2y_2+y_3=n-3}\binom{y_1+y_2+y_3}{y_1,\ y_2,\ y_3}3^{y_2+y_3}D^{y_2+2y_3}}{\sum\limits_{3y_1+2y_2+y_3=n-1}\binom{y_1+y_2+y_3}{y_1,\ y_2,\ y_3}3^{y_2+y_3}D^{y_2+2y_3}}\right. \\ &\quad + 2D\frac{\sum\limits_{3y_1+2y_2+y_3=n-2}\binom{y_1+y_2+y_3}{y_1,\ y_2,\ y_3}3^{y_2+y_3}D^{y_2+2y_3}}{\sum\limits_{3y_1+2y_2+y_3=n-1}\binom{y_1+y_2+y_3}{y_1,\ y_2,\ y_3}3^{y_2+y_3}D^{y_2+2y_3}} \end{aligned}\right\}$$

This limiting expression for w^2 takes on a special simple form for D=1.

For D=1 the matricians, which were calculated previously, became

$$A_0^{(3)} = 1, A_0^{(4)} = 3, A_0^{(5)} = 12, A_0^{(6)} = 46, A_0^{(7)} = 177, A_0^{(8)} = 681, A_0^{(9)} = 2620.$$

With those values we obtain from the formula (4*.11)

$$\sqrt[3]{2} = 1 + \lim_{n\to\infty}\frac{A_0^{(n)}}{A_0^{(n+1)}},$$

$$\sqrt[3]{2} \approx 1+\frac{1}{3}, 1+\frac{3}{12}, 1+\frac{12}{46}, 1+\frac{46}{177}, 1+\frac{177}{681}, 1+\frac{681}{2620},$$

$$\sqrt[3]{2} \approx 1.3, 1.25, 1.26, 1.26, 1.26, 1.26.$$

Thus $\sqrt[3]{2} = 1.26$ is quite an exact approximation. We have

$$2 \approx 1.26^3 = 2.000376$$

We further obtain from the first line of (5*.3) with D=1

$$\sqrt[3]{4} \approx 1 + \frac{A_0^{(n)}}{A_0^{(n+2)}} + \frac{2A_0^{(n+1)}}{A_0^{(n+2)}},$$

$$\sqrt[3]{4} \approx 1+\frac{1}{12}+\frac{1}{2},\ 1+\frac{3}{16}+\frac{24}{46},\ 1+\frac{12}{177}+\frac{92}{177},$$

$$1+\frac{46}{681}+\frac{354}{681}, 1+\frac{177}{2620}+\frac{1362}{2620}.$$

$\sqrt[3]{4} = 1.58, 1.59, 1.59, 1.59, 1.58.$ Thus $\sqrt[3]{4} = 1.59$ is a satisfying approximation.

For $D = 2, w^2 = \sqrt[3]{81} = 3\sqrt[3]{3}.$

Thus also $\sqrt{3}$ can be easily and perfectly approximated by formula (4*.11).

6*. **The case n=5**

We denote again

$$w = \sqrt[5]{D^5 + 1} \tag{6*.1}$$

and choose the fixed vector

$$a^{(0)} = (w + 4D, w^2 + 3wD + 6D^2, w^3 + 2w^2D + 4D^3, w^4 + Dw^3 + D^2w^2 + D^3w + D^4) \tag{6*.2}$$

With $a^{(0)} = (a_1^{(0)}(w), a_2^{(0)}(w), a_3^{(0)}(w), a_4^{(0)}(w))$, we again apply the rule for calculating the components of the companion vectors

$$b_i^{(v)} = a_i^{(v)}(D), \; i = 1, 2, 3, 4; \; v = 0, 1, ...$$

We proceed with the (BGEA) of a $a^{(0)}$ and have

$$b^{(0)} = (D + 4D, \; D^2 + 3D^2 + 6D^2, \; D^3 + 2D^3 + 3D^3 + 4D^3, \; D^4 + D^4 + D^4 + D^4 + D^4)$$

$$\left.\begin{aligned} b^{(0)} &= (5D, 10D^2, 10D^3, 5D^4) \\ b^{(0)} &= \left(\binom{5}{1}D, \binom{5}{2}D^2, \binom{5}{3}D^3, \binom{5}{4}D^4\right) \end{aligned}\right\} \tag{6*.3}$$

$$a^{(1)} = (w + 4D - 5D)^{-1} \cdot (w^2 + 3wD + 6D^2 - 10D^2,$$
$$w^3 + 2w^2D + 3wD^2 + 4D^3 - 10D^3, \; w^4 + Dw^3 + D^2w^2 + D^3w + D^4 - 5D^4, 1),$$

$$a^{(1)} = (w - D)^{-1} \cdot ((w - D)(w + 4D), (w - D)(w^2 + 3wD + 6D^2),$$
$$(w - D)(w^3 + 2w^2D + 3wD^2 + 4D^3, \; (w - D)(w^4 + w^3D + w^2D^2 + wD^3 + D^4)),$$

$$a^{(1)} = a^{(0)}. \tag{6*.4}$$

Since $(w - D)(w^3 + w^3D + w^2D^2 + wD^3 + D^4) = w^5 - D^5 = 1.$ Thus the (BGEA) of $a^{(0)}$ is purely periodic with lengths of primitive period *l*=1. We shall calculate the matricians of $a^{(0)}$.

With (4*.6) we have

$$\left.\begin{aligned} &A_0^{(0)} = 1, A_0^{(1)} = A_0^{(2)} = A_0^{(3)} = A_0^{(4)} = 0 \\ &A_0^{(n+5)} = A_0^{(n)} + 5DA_0^{(n+1)} + 10D^2A_0^{(n+2)} + 10D^3A_0^{(n+3)} + 5D^4A_0^{(n+4)}, \\ &n = 0, 1, ... \end{aligned}\right\} \tag{6*.5}$$

Proceeding as in case n=2,3 using Euler's functions, we obtain

$$\sum_{n=0}^{\infty} A_0^{(n)} x^n = 1 + \sum_{n=0}^{\infty} A_0^{(n+5)} x^{n+5} =$$

$$= 1 + \sum_{n=0}^{\infty} (A_0^{(n)} + 5DA_0^{(n+1)} + 10D^2 A_0^{(n+2)} + 10D^3 A_0^{(n+3)} + 5D^4 A_0^{(n+4)}) \cdot x^{n+5}$$

$$= x^5 \sum_{n=0}^{\infty} A_0^{(n)} x^n + 5D(-1 + \sum_{n=0}^{\infty} A_0^{(n)} x^n) x^4 +$$

$$+ 10D^3 x^3 (-1 + \sum_{n=0}^{\infty} A_0^{(n)} x^n) + 10D^3 x^2 (-1 + \sum_{n=0}^{\infty} A_0^{(n)} x^n) + 5D^4 x(-1 + \sum_{n=0}^{\infty} A^{(n)} x^n);$$

$$[1 - (x^5 + 5Dx^4 + 10D^2 x^3 + 10D^3 x^2 + 5D^4 x)] \sum_{n=0}^{\infty} A^{(n)} x^{(n)} =$$

$$= -(5Dx^4 + 10D^2 x^3 + 10D^3 x^2 + 5D^4 x),$$

$$\sum_{n=0}^{\infty} A_0^{(n+5)} x^n = \frac{1}{1 - (x^5 + 5Dx^4 + 10D^2 x^3 + 10D^3 x^2 + 5D^4 x)} =$$

$$= \sum_{k=o}^{\infty} (x^5 + 5Dx^4 + 10D^2 x^3 + 10D^3 x^2 + 5D^4 x).$$

Hence

$$A_0^{(n+5)} = \sum_{5y_1+4y_2+3y_3+2y_4+y_5=n} \binom{y_1+y_2+y_3+y_4+y_5}{y_1,\ y_2,\ y_3,\ y_{4,5}\, y_5} (5D)^{y_2} (10D^2)^{y_3} (10D^3)^{y_4} (5D^4)^{y_5}$$

$$= \sum_{5y_1+4y_2+3y_3+2y_4+y_5=n} \binom{y_1+y_2+y_3+y_4+y_5}{y_1,\ y_2,\ y_3,\ y_4,\ y_5} 5^{y_2+y_3+y_4+y_5} 2^{y_3+y_4} D^{y_2+2y_3+3y_4+4y_5}$$

(6*.6)

$$\left. \begin{aligned} A_0^{(n+5)} &= \sum_{\sum_{i=0}^{4}(5-i)y_{i+1}=n} \binom{\sum_{k=1}^{5} y_k}{y_1, y_2, y_3, y_4, y_5} 5^{\sum_{k=1}^{4} y_{k+1}} 2^{y_3+y_4} D^{y_2+2y_3+3y_4+4y_5,} \\ & n = 0,1,\ldots \end{aligned} \right\}$$

Formula (6*.6) is, indeed a bit frightening, but it calculates explicitly any $A_0^{(n+5)}$ (n=0,1,…) just by solving the linear equations $\sum_{i=o}^{4} (5-i) y_{i+1} = n$.

We shall calculate a few matricians from formula (6*.6). When one of the y_i-s will not be mentioned, its value is understood to equal zero.

$$n = 1; y_5 = 1, A_0^{(6)} = 5D^4;$$

$$n = 2;\ i)\ y_4 = 1;\ ii)\ y_5 = 2;\ A_0^{(7)} = 10D^3 + 25D^8;$$

$$n = 3;\ i)\ y_3 = 1;\ ii)\ y_4 = y_5 = 1;\ iii)\ y_5 = 3,$$

$$A_0^{(8)} = 10D^2 + 100D^7 + 125D^{12};$$

$$n = 4;\ i)\ y_2 = 1;\ ii)\ y_3 = y_5 = 1;\ iii)\ y_4 = 2,\ iv)\ y_4 = 1, y_5 = 2;$$

$$v)\ y_5 = 4,\ A_0^{(9)} = 5D + 100D^6 + 100D^6 + 750D^{11} + 625D^{16},$$

$$A_0^{(9)} = 5D + 200D^6 + 750D^{11} + 625D^{16};$$

$$n = 5;\ i)\ y_1 = 1;\ ii)\ y_2 = y_3 = 1;\ iii)\ y_3 = y_4 = 1,\ iv)\ y_3 = 1, y_5 = 2;$$

$$v)\ y_4 = 2,\ y_5 = 1; vi)\ y_4 = 1,\ y_5 = 3; vii)\ y_5 = 5,$$

$$A_0^{(10)} = 1 + 50D^5 + 200D^5 + 750D^{10} + 1500D^{10} + 5000D^{15} + 3125D^{20}.$$

$$A_0^{(10)} = 1 + 250D^5 + 2250D^{10} + 5000D^{15} + 3125D^{20}.$$

We shall further calculate the matricians $A_i^{(v)},\ i = 1,2,3,4.$ We have

$$\left.\begin{aligned} &A_1^{(n)} + (w + 4D)A_1^{(n+1)} + (w^2 + 3Dw + 6D^2)A_1^{(n+2)} + \\ &\quad + (w^3 + 2w^2D + 3wD^2 + 4D^3)A_1^{(n+3)} + \\ &\quad + (w^4 + Dw^3 + D^2w^2 + D^3w + D^4)A_1^{(n+4)} = \\ &\quad = (w + 4D)A_0^{(n)} + (w + 4D)A_0^{(n+1)} + (w^2 + 3Dw + 6D^2)A_0^{(n+2)} + \\ &\quad + (w^3 + 2w^2D + 3wD^2 + 4D^3)A_0^{(n+3)} + \\ &\quad + (w^4 + Dw^3 + D^2w^2 + D^3w + D^4)A_0^{(n+4)}. \end{aligned}\right\} \tag{6*.7}$$

Comparing powers of w^4 on both sides of (6*.7) we obtained

$$A_1^{(n+4)} = A_0^{(n+3)} + 5DA_0^{(n+4)}. \tag{6*.8}$$

Further

$$A_2^{(n)} + (w + 4D)A_2^{(n+1)} + ... + (w^4 + ... + D^4)A_2^{(n+4)} =$$

$$= (w^2 + 3Dw + 6D^2)[A_0^{(n)} + ... + (w^2 + 3Dw + 6D^2)A_0^{(n+2)}$$

$$+ (w^3 + 2w^2D + 3Dw^2 + 4D^3)A_0^{(n+3)} + (w^4 + Dw^3 + D^2w^2 + D^3w + D^4)A_0^{(n+2)}].$$

$$A_2^{(n+4)} = A_0^{(n+2)} + 5DA_0^{(n+3)} + 10D^2A_0^{(n+4)} \tag{6*.9}$$

We leave it to the reader to verify, by the same method the results

$$A_3^{(n+4)} = A_0^{(n+1)} + 5DA_0^{(n+2)} + 10D^2A_0^{(n+3)} + 10D^3A_0^{(n+4)} \tag{6*.10}$$

(6*.11) $$A_4^{(n+4)} = A_0^{(n)} + 5DA_0^{(n+1)} + 10D^2A_0^{(n+2)} + 10D^3A_0^{(n+3)} + 5D^4A_0^{(n+4)}.$$

We thus have obtained

$$A_1^{(n+4)} = A_0^{(n+3)} + \binom{5}{1}A_0^{(n+4)}$$

$$A_2^{(n+4)} = A_0^{(n+2)} + \binom{5}{1}A_0^{(n+3)} + \binom{5}{2}A_0^{(n+4)}$$

$$A_3^{(n+4)} = A_0^{(n+1)} + \binom{5}{1}DA_0^{(n+2)} + \binom{5}{2}D^2A_0^{(n+3)} + \binom{5}{3}D^3A_0^{(n+4)}$$

$$A_4^{(n+4)} = A_0^{(n)} + \binom{5}{1}DA_0^{(n+1)} + \binom{5}{2}D^2A_0^{(n+2)} + \binom{5}{3}D^3A_0^{(n+3)} + \binom{5}{4}D^4A_0^{(n+4)}$$

(6*.12) $$A_i^{(n+4)} = \sum_{j=0}^{i} A_0^{(n+4-j)}\binom{5}{i-j}D^{i-j},\ \ i = 1,2,3,4,\ n = 0,1,...$$

To conclude with this paper of the sequence, we shall still approximate the number $\sqrt[5]{2}, \sqrt[5]{4}$. From (6*.1) we have, with D=1, $w = \sqrt[5]{2}$.
From (6*.2) we obtain, with Theorem 1, and from (6*.8)

(6*.13) $$w + 4 = \lim_{n\to\infty}\frac{A_1^{(n+6)}}{A_0^{(n+6)}},\ \ w + 4 = \lim_{n\to\infty}\frac{A_1^{(n+6)}}{A_0^{(n+6)}},\ \ w = 1 + \lim_{n\to\infty}\frac{A_0^{(n+5)}}{A_0^{(n+6)}}$$

From (6*.14) and (6*.6) we finally obtain

(6*.15) $$\sqrt[5]{2} \approx 1 + \frac{\displaystyle\sum_{5y_1+4y_2+3y_3+2y_4+y_5=n}\binom{y_1+y_2+y_3+y_4+y_5}{y_1,\ y_2,\ y_3,\ y_4,\ y_5}5^{(y_2+y_3+y_4+y_5)}2^{(y_3+y_4)}D^{(y_2+2y_3+3y_4+4y_5)}}{\displaystyle\sum_{5y_1+4y_2+3y_3+2y_4+y_5=n+1}\binom{y_1+y_2+y_3+y_4+y_5}{y_1,\ y_2,\ y_3,\ y_4,\ y_5}5^{(y_2+y_3+y_4+y_5)}2^{(y_3+y_4)}D^{(y_2+2y_3+3y_4+4y_5)}}$$

Substituting in (6*.14) the values for n=1,2,3,4 we obtain

$$w = \sqrt[5]{2} \approx 1 + \frac{5}{35},\ 1 + \frac{35}{235},\ 1 + \frac{235}{1580},\ 1 + \frac{1580}{10626},$$

$$\sqrt[5]{2} \approx 1.143,\ 1.149,\ 1.149,\ 1.149.$$

It seems that 1.149 is a good approximation of $\sqrt[5]{2}$. Indeed, $1.149^2 \approx 2$.
From (6*.2) and Theorem 1 we further obtain, in virtue of (6*.9)

$$w^2 + 3wD + 6D^2 = \lim_{n\to\infty} \frac{A_2^{(n+4)}}{A_0^{(n+4)}} = 10D^2 + 5D \lim_{n\to\infty} \frac{A_0^{(n+3)}}{A_0^{(n+4)}} + \lim_{n\to\infty} \frac{A_0^{(n+2)}}{A_0^{(n+4)}},$$

$$w^2 + 3D\left(D + \lim_{n\to\infty} \frac{A_0^{(n+3)}}{A_0^{(n+4)}}\right) + 6D^2 = 10D^2 + 5D \lim_{n\to\infty} \frac{A_0^{(n+3)}}{A_0^{(n+4)}} + \lim_{n\to\infty} \frac{A_0^{(n+2)}}{A_0^{(n+4)}},$$

Hence

$$w^2 = D^2 + 2D \lim_{n\to\infty} \frac{A_0^{(n+3)}}{A_0^{(n+4)}} + \lim_{n\to\infty} \frac{A_0^{(n+2)}}{A_0^{(n+4)}}$$

with D=1, and the approximate value of w, we obtain.

$$w^2 \approx 1 + 2 \cdot 0{,}149 + \frac{A_0^{(n+2)}}{A_0^{(n+4)}} \quad \text{or} \quad w^2 \approx 1.298 + \frac{A_0^{(n+2)}}{A_0^{(n+4)}}$$

From here w^2 can be easily evaluated exact to two places.

Section 2. Some new combinatorial identities derived from units in algebraic number fields

Two new combinatorial identities are derived from explicitly states units in algebraic number fields of degree n=3. Let $e = w - D$ be a units in a cubic field where $w^3 = D^3 + 1, D \in N$ and n=0,1,... Then $z_n = t_{n+1}^2 - t_n t_{n+2}$ and $t_n = z_{n-1}^2 - z_{n-2} z_n$ are two combinatorial identities where t_{n+2} and z_{z+1} are obtained from some recursion formula. Both have the same structure.

1*. **Introduction**.

One could be surprised why such a mathematical field as combinatorics should be coupled with the units in algebraic number fields. The connection between these two, seemingly different domains of mathematics, rests upon the fact that Bernstein invented a method from which combinatorial identities can be derived from explicitly stated units in algebraic number fields of any degree n• 2. It is not really necessary to make use of the theory of units to produce such results since Carlitz, who is the master of these results, obtained Bernstein's combinatorial identities and many other more difficult ones using the classical method. This method is interesting since, sometimes Bernstein's method is capable of producing some results with simpler computations. The difference between the combinatorial identities established by Bernstein and those generated in this section rests, of course, with the choice of units.

We choose our units from the set of units given by

$$e=\frac{(w-D)^s}{w^s-D^s},\ 1<s\leq n, w^n=D^n+d,\ d|D, D\in N,\ d\in Z,\ s|n.$$

The author derived these units from the periodicity of (BGEA) developed before.

THEOREM 1. Let $e=w-D$ be a unit in a cubic field where $w^3=D^3+1$, $D\in N$, and n=0,1,…Then

$$z_n=t_{n+1}^2-t_n t_{n+2} \quad \text{and} \quad t_n=z_{n-1}^2-z_{n-2}z_n$$

are two combinatorial identities where t_{n+2} and z_{n+1} are as follows:

$$t_{n+2}=\sum_{\substack{x_1+x_2+x_3=i\\3x_1+2x_2+x_3=n}}(-1)^{x_2+x_3}\binom{i}{x_1,x_2,x_3}3^{x_2+x_3}D^{2x_2+x_3}$$

and

$$z_{n+1}=\sum_{\substack{y_1+y_2+y_3=i\\3y_1+2y_2+y_3=n}}\binom{i}{y_1,y_2,y_3}3^{y_2+y_3}D^{y_2+2y_3}$$

2*. **Positive powers of e**

In this section we shall calculate positive powers of e=w-D, taking in consideration that $w^3=D^3+1$:

$$e^0=1=1+0\cdot w+0\cdot w^2,$$

$$e=w-D=-D+1\cdot w+0\cdot w^2,$$

$$e^2=(w-D)^2=D^2-2Dw+1\cdot w^2,$$

$$e^3=(w-D)^3=1+3D^2w-3Dw^2,$$

$$e^4=(w-D)^4=(-3D^4-4D)-(3D^3-1)w+6D^2w^2,$$

$$e^5=(w-D)^5=9D^5+10D^2-5Dw-(9D^3-1)w^2.$$

We now denote

(2*.1) $$e^n=(w-D)^n=r_n+s_nw+t_nw^2,\ r_n,s_n,t_n\in Z, n=0,1,...,$$

From our previous calculations we have

(2*.2)
$$\begin{array}{lll} r_0 = 1, & s_0 = 0, & t_0 = 0, \\ r_1 = D, & s_1 = 1' & t_1 = 0, \\ r_2 = D^2, & s_2 = -2D, & t_2 = 1, \\ r_3 = 1, & s_3 = 3D^2, & t_3 = -3D, \\ r_4 = -3D^4 - 4D, & s_4 = -3D^3 + 1, & t_4 = 6D^2, \\ r_5 = 9D^5 + 10D^2, & s_5 = -5D, & t_5 = -9D^3 + 1. \end{array}$$

We denote

(2*.3)
$$w^3 = D^3 + 1 = m$$

and multiply both sides of (2*.1) by e = w-D, we obtain

(2*.4)
$$\begin{aligned} e^{n+1} &= r_n (w - D) + s_n (w^2 - Dw) + t_n (w^3 - Dw^2) \\ &= mt_n - Dr_n + (r_n - Ds_n)w + (s_n - Dt_n)w^2. \end{aligned}$$

From (2*.1) and (2*.4) we have

(2*.5a)
$$r_{n+1} = mt_n - Dr_n,$$

(2*.5b)
$$s_{n+1} = r_n - Ds_n,$$

(2*.5c)
$$t_{n+1} = s_n - Dt_n.$$

From (2*.5c) we obtain

(2*.6)
$$s_n = t_{n+1} + Dt_n.$$

From (2*.5b) and (2*.6) we obtain

(2*.7)
$$r_n = t_{n+2} + 2Dt_{n+1} + D^2 t_n.$$

From (2*.5a) and (2*.7) we obtain

$$\begin{aligned} t_{n+3} + 2Dt_{n+2} + D^2 t_{n+1} &= mt_n - D(t_{n+2} + 2Dt_{n+1} + D^2 t_n) \\ &= (D^3 + 1)t_n - Dt_{n+2} - 2D^2 t_{n+1} - D^3 t_n \end{aligned}$$

(2*.8)
$$t_{n+3} = t_n - 3D^2 t_{n+1} - 3Dt_{n+2}.$$

Substituting the values of s_n, r_n, from (2*.6), (2*.7) in (2*.1)

(2*.9)
$$\begin{aligned} e^n &= t_{n+2} + 2Dt_{n+1} + D^2 t_n + (t_{n+1} + Dt_n)w + t_n w^2, \\ t_{n+3} &= t_n - 3D^2 t_{n+1} - 3Dt_{n+2} \end{aligned}$$

(2*.9) expresses e^n by one parameter t_n recursively given and w. For t_0=t_1, t_2=1 as in (2*.2) we calculate t_n by using Euler's generating function:

$$\sum_{n=0}^{\infty} t_n u^n = t_o u_o + t_1 u + t_2 u^2 + \sum_{n=3}^{\infty} t_n u^n = u^2 + \sum_{n=0}^{\infty} t_{n+3} u^{n+3}$$

$$= u^2 + u^3 \sum_{n=0}^{\infty} (t_n - 3D^2 t_{n+1} - 3Dt_{n+2}) u^n$$

$$= u^2 + u^3 \sum_{n=0}^{\infty} t_n u^n - 3D^2 u^2 \sum_{n=0}^{\infty} t_{n+1} u^{n+1} - 3Du \sum_{n=0}^{\infty} t_{n+2} u^{n+2}$$

$$= u^2 + u^3 \sum_{n=0}^{\infty} t_n u^n - 3D^2 u^2 \left[\left(\sum_{n=0}^{\infty} t_n u^n \right) - t_0 u^0 \right] - 3Du \left[\left(\sum_{n=0}^{\infty} t_n u^n \right) - t_o u^0 - t_1 u \right],$$

and since $t_0 = t_1 = 0$,

$$\sum_{n=0}^{\infty} t_n u^n = u^2 + u^3 \sum_{n=0}^{\infty} t_n u^n - 3D^2 u^2 \sum_{n=0}^{\infty} t_n u^n - 3Du \sum_{n=0}^{\infty} t_n u^n,$$

$$(1 - u^3 + 3D^2 u^2 + 3Du) \sum_{n=0}^{\infty} t_n u^n = u^2,$$

$$\sum_{n=0}^{\infty} t_n u^n = \frac{u^2}{1 - (u^3 - 3D^2 u^2 - 3Du)},$$

$$t_o u^0 + t_1 u + \sum_{n=2}^{\infty} t_n u^n = \frac{u^2}{1 - (u^3 - 3D^2 u^2 - 3Du)}$$

$$\sum_{n=0}^{\infty} t_{n+2} u^{n+2} = \frac{u^2}{1 - (u^3 - 3D^2 u^2 - 3Du)},$$

(2*.10) $$\sum_{n=0}^{\infty} t_{n+2} u^n = \frac{1}{1 - (u^3 - 3D^2 u^2 - 3Du)}.$$

We choose u sufficiently small such that

(2*.11) $$\left| u^3 - 3D^2 u^2 - 3Du \right| < 1.$$

It suffices to choose $|u| < 1/9D$

$$\left| u^3 - 3D^2 u^2 - 3Du \right| \le \left| u^3 \right| + 3D^2 \left| u^2 \right| + 3D|u|$$

$$< \frac{1}{729D^3} + \frac{3D^2}{81D^3} + \frac{3D}{9D} < \frac{1}{3D} + \frac{1}{3D} + \frac{1}{3} < 1.$$

Since D>1 we expressed the right-hand side of (2*.10) as an infinite, absolutely convergent series and obtain:

(2*.12) $$\sum_{n=0}^{\infty} t_{n+2} u^n = \sum_{j=0}^{\infty} (u^3 - 3D^2 u^2 - 3Du)^j.$$

We find t_{n+2} (n=1,2,…) from (2*.12) by comparison of coefficients. Now, we are looking for the coefficients of u^n in the expansion on the right-hand side. Here the exponent j varies from 0 to ∞. Since the polynomial which is being raised to the power j has the highest power u^3, the exponent j, in order to obtain u^n cannot be smaller than $\left[\frac{1}{3}n\right]$ and since this polynomial has the smallest power u, the exponent j cannot be larger than n. Thus in order to obtain all powers of u^n in the expansion of the right-hand side of (2*.12) we have to investigate the sum

(2*.13) $$\sum_{i=[n/3]}^{n}(u^3 - 3D^2u^2 - 3Du)^i.$$

We expand the polynomial in (2*.13) by the multinomial formula and obtain

(2*.14) $$(u^3 - 3D^2u^2 - 3Du)^i = \sum_{x_1+x_2+x_3=i}\binom{i}{x_1, x_2, x_3}(u^3)^{x_1}(-3D^2u^2)^{x_2}(-3Du)^{x_3}$$

or

(2*.15)

$$(u^3-3D^2u^2-3Du)^i = \sum_{x_1+x_2+x_3=i}\binom{i}{x_1, x_2, x_3}(-1)^{x_2+x_3}3^{x_2+x_3}D^{2x_2+x_3}u^{3x_1-2x_2+x_3}.$$

now we are looking for elements u^n and we have to set

(2*.16) $$3x_1 + 2x_2 + x_3 = n$$

From the two conditions

(2*.17) $$x_1 + x_2 + x_3 = i, \quad 3x_1 + 2x_2 + x_3 = n,$$

and the bounds of i, we finally have

(2*.18) $$t_{n+2} = \sum_{i=[n/3]}^{n}\sum_{\substack{x_1+x_2+x_3=i \\ 3x_1+2x_2+x_3=n}}(-1)^{x_2+x_3}\binom{i}{x_1, x_2, x_3}3^{x_2+x_3}D^{2x_2+x_3}$$

(2*.18) is the formula which states t_n in an explicit form.

The bounds of i are already given by the two restriction (2*.17), so that formula (2*.18) can also be written as

(2*.19) $$t_{n+2} = \sum_{\substack{x_1+x_2+x_3=i \\ 3x_1+2x_2+x_3=n}}(-1)^{x_2+x_3}\binom{i}{x_1, x_2, x_3}3^{x_2+x_3}D^{2x_2+x_3}$$

3*. **Negative powers of e**

This section is devoted to the calculation of the negative powers of e. Since this problem is essentially the same as the finding of the positive powers in the previous section, we shall carry out many operations without giving again the necessary explanations. We shall set out with calculating a few initial values of e^{-1}, with e=w-D, $w^3=D^3+1$, we have $w^3-D^3=1$, $(w-D)(w^2+Dw+D^2)=1$.

(3*.1) $$e^{-1}=(w-D)^{-1}=\frac{1}{w-D}=D^2+Dw+w^2,$$

$$e^{-2}=(D^2+Dw+w^2)^2=3D^4+2D+(3D^3+1)w+3D^2w^2.$$

Denoting

(3*.2) $$e^{-n}=x_n+y_nw+z_nw^2,\ \ n=0,1,\dots,x_n,y_n,z_n\in Z,\ \text{for } n>0,$$

we have for the initial values

$$x_o=1,\qquad y_o=0,\qquad z_0=0,$$

(3*.3) $$x_1=D^2,\qquad y_1=D,\qquad z_1=1,$$

$$x_2=3D^4+2D,\ y_2=3D^3+1,\ z_2=3D^2.$$

From (3*.2) we obtain, multiplying both sides by $e^{-1}=D^2+Dw+w^2$,

$$e^{-(n+1)}=D^2x_n+(D^3+1)y_n+D(D^3+1)z_n$$
$$+[Dx_n+D^2y_n+(D^3+1)z_n]w+(x_n+Dy_n+D^2z_n)w^2.$$

Hence by definition

(3*.4a) $$x_{n+1}=D^2x_n+(D^3+1)y_n+D(D^3+1)z_n,$$

(3*.4b) $$y_{n+1}=Dx_n+D^2y_n+(D^3+1)z_n,$$

(3*.4c) $$z_{n+1}=x_n+Dy_n+D^2z_n.$$

From (3*.4) we obtain, making the operations (3*.4a)-D(3*.4b), (3*.4b)-D(3*.4c),

(3*.5) $$x_{n+1}-Dy_{n+1}=y_n,\quad y_{n+1}-Dz_{n+1}=z_n,$$

From (3*.5) $$x_{n+1}=y_n+Dy_{n+1},\quad x_n=y_{n+1}+Dy_n,$$

(3*.6) $$y_{n+1}=z_n+Dz_{n+1},\quad x_n=z_{n-2}+2Dz_{n-1}+D^2z_n.$$

From (3*.4c) we obtain the recursion formula

$$z_{n+1}=z_{n-2}+3Dz_{n-1}+3D^2z_n\quad\text{or}$$

(3*.7) $$z_{n+3}=z_n+3Dz_{n+1}+3D^2z_{n+2},$$

(3*.8) $$e^{-n}=z_{n-2}+2Dz_{n-1}+D^2z_n+(z_{n-1}+Dz_n)w+z_nw^2.$$

With the recursion formula (3*.7) we can calculate z_n explicitly by Euler's generating function, viz.

(3*.9) $$\sum_{n=0}^{\infty} z_n u^n = z_0 + z_1 u + z_2 u^2 + \sum_{n=3}^{\infty} z_n u^n .$$

Substituting in (3*.9) the first initial values of z_n from the table (3*.3) we obtain

$$\sum_{n=0}^{\infty} z_n u^n = u + 3D^2 u^2 + \sum_{n=0}^{\infty} z_{n+3} u^{n+3}$$

$$= u + 3D^2 u^2 + u^3 \sum_{n=0}^{\infty} (z_n + 3Dz_{n+1} + 3D^2 z_{n+2}) u^n$$

$$= u + 3D^2 u^2 + u^3 \sum_{n=0}^{\infty} z_n u^n + 3Du^2 \sum_{n=0}^{\infty} z_{n+1} u^{n+1} + 3D^2 u \sum_{n=0}^{\infty} z_{n+2} u^{n+2}$$

$$= u + 3D^2 u^2 + u^3 \sum_{n=0}^{\infty} z_n u^n + 3Du^2 \left[\left(\sum_{n=0}^{\infty} z_n u^n \right) - z_0 u^0 \right] =$$

$$+ 3D^2 u \left[\left(\sum_{n=0}^{\infty} z_n u^n \right) - z_0 u^0 - z_1 u \right]$$

$$= u + 3D^2 u^2 + u^3 \sum_{n=0}^{\infty} z_n u^n + 3Du^2 \sum_{n=0}^{\infty} z_n u^n + 3D^2 u \sum_{n=0}^{\infty} z_n u^n - 3D^2 u^2$$

$$= u + u^3 \sum_{n=0}^{\infty} z_n u^n + 3Du^2 \sum_{n=0}^{\infty} z_n u^n + 3D^2 u \sum_{n=0}^{\infty} z_n u^n .$$

Thus

$$[1 - (u^3 + 3Du^2 + 3D^2 u)] \sum_{n=0}^{\infty} z_n u^n = u,$$

$$\sum_{n=0}^{\infty} z_n u^n = \frac{u}{1 - (u^3 + 3Du^2 + 3D^2 u)},$$

$$z_0 u^0 + \sum_{n=0}^{\infty} z_{n+1} u^{n+1} = \frac{u}{1 - (u^3 + 3Du^2 + 3D^2 u)},$$

(3*.10) $$\sum_{n=0}^{\infty} z_{n+1} u^n = \frac{1}{1 - (u^3 + 3Du^2 + 3D^2 u)} .$$

Choosing $u<1/9D^2$ we have $\left| u^3 + 3Du^2 + 3D^2 u \right| < 1$ so that from (3*.10) we obtain

(3*.11) $$\sum_{n=0}^{\infty} z_{n+1}u^n = \sum_{i=0}^{\infty}(u^3 + 3Du^2 + 3D^2u)^i$$

Using the multinomial formula we obtain

$$(u^3 + 3Du^2 + 3D^2u)^i = \sum_{y_1+y_2+y_3=i} \binom{i}{y_1, y_2, y_3} u^{3y_1}(3Du^2)^{y_2}(3D^2u)^{y_3},$$

$$(u^3 + 3Du^2 + 3D^2u)^i = \sum_{y_1+y_2+y_3=i} \binom{i}{y_1, y_2, y_3} 3^{y_2+y_3} D^{y_2+2y_3} u^{3y_1+2y_2+y_3}.$$

To find the coefficient of u^n in the expansion on the right side of (3*.11) we obtain

(3*.12) $$z_{n+1} = \sum_{\substack{y_1+y_2+y_3=i \\ 3y_1+2y_2+y_3=n}} \binom{i}{y_1, y_2, y_3} 3^{y_2+y_3} D^{y_2+2y_3}.$$

The values of *i* in (3*.12) are determined by two diophantine equations

(3*.13) $$y_1 + y_2 + y_3 = i,\ 3y_1 + 2y_2 + y_3 = n, \quad n \geq 0,\ \ y_1, y_2, y_3 \geq 0;\ \left[\frac{1}{3}n\right] \leq i \leq n.$$

4*. **New combinatorial identities**

We have

(4*.1) $$e^n e^{-n} = (r_n + s_n w + t_n w^2)(x_n + y_n w + z_n w^2) = 1$$

Multiplying on the right-hand side of (4*.1) and substituting $w^3 = D^3 + 1 = m, w^4 = mw,$ we obtain

(4*.12) $$\begin{aligned} 1 = r_n x_n + r_n y_n w + r_n z_n w^2 + ms_n z_n + s_n x_n w \\ + s_n y_n w^2 + mt_n y_n + mt_n z_n w + t_n x_n w^2 \end{aligned}$$

From (4*.2) we obtain, by comparison of coefficients of equal powers of w (and because w is a third degree algebraic irrational)

(4*.3) $$\begin{aligned} r_n x_n + mt_n y_n + ms_n z_n &= 1, \\ s_n x_n + r_n y_n + mt_n z_n &= 0, \\ t_n x_n + s_n y_n + r_n z_n &= 0. \end{aligned}$$

In (4*.3) we consider x_n, y_n, z_n as indeterminates, and r_n, s_n, t_n as coefficiential factors and

(4*.4) $$\Delta = \begin{vmatrix} r_n & mt_n & ms_n \\ s_n & r_n & mt_n \\ t_n & s_n & r_n \end{vmatrix}$$

The determinant (4*.4) is the norm of e^n. We have

$$e^n = r_n + s_n w + t_n w^2,$$
(4*.5)
$$e^n w = mt_n + r_n w + s_n w^2,$$
$$e^n w^2 = ms_n + mt_n w + r_n w^2.$$

From (4*.5) we obtain

(4*.6)
$$N(e^n) = \begin{vmatrix} r_n & s_n & t_n \\ mt_n & r_n & s_n \\ ms_n & mt_n & r_n \end{vmatrix} = \begin{vmatrix} r_n & mt_n & ms_n \\ s_n & r_n & mt_n \\ t_n & s_n & r_n \end{vmatrix} = \Delta$$

We shall now calculate $N(e^n)$.

(4*.7)
$$N(e^n) = [N(e)]^n,$$

(4*.8)
$$N(e) = N(w - D) = N[-(D - w)] = (-1)^3 N(D - w) = -N(D - w).$$

(4*.9)
$$D^3 + 1 = w^3,\ D^3 - w^3 = -1$$
$$(D - w)(D - \rho w)(D - \rho^2 w) = -1,\ \rho = e^{2\pi i/3}$$

We thus obtain $N(e) = -N(D - w) = -1(-1) = 1$.

(4*.10)
$$N(e) = N(e^n) = 1 = \Delta$$

From (4*.3), by Cramer's rule and Ä=1 we have

(4*.11)
$$z_n = s_n^2 - r_n t_n.$$

Substituting in (4*.11) the values for s_n and r_n from (2*.9) we obtain

$$z_n = (t_{n+1} + Dt_n)^2 - (t_{n+2} + 2Dt_{n+1} + D^2 t_n)t_n \quad \text{or}$$

(4*.12)
$$z_n = t_{n+1}^2 - t_n t_{n+2},\ n = 0,1,...$$

Substituting in (4*.12) the values of z_n from (3*.12) and t_n from (2*.12), we obtain the first wanted combinatorial identities.

We return to the system equation (4*.3) and rearrange it, considering r_n, s_n, t_n, as indeterminates where the determinant of the equations this time equals

(4*.13)
$$\Delta_1 = \begin{vmatrix} x_n & mz_n & my_n \\ y_n & x_n & mz_n \\ z_n & y_n & x_n \end{vmatrix}.$$

As before $\Delta_1 = N(e^{-n}) = (N(e))^{-1} = 1^{-1} = 1$. In a similar way we obtain

(4*.14)
$$t_n = y_n^2 - x_n z_n.$$

Substituting in (4*.14) the values of x_n, y_n from (3*.8) we obtain

(4*.15) $t_n = (z_{n-1} + Dz_n)^2 - z_n(z_{n-2} + 2Dz_{n-1} + D^2 z_n)$ or $t_n = z_{n-1}^2 - z_{n-2} z_n$.

(4*.15) supplies a second combinatorial identity. Both have the same structure.

Section 3. Some diophantine equations and identities from (BGEA)

In this section we shall investigate Diophantine equations of the second and third degree of a special type. The general equations of the second and third degrees are far from being totally solved. It suffices to look up Mordell's book on Diophantine equations, to learn how little we actually know about the general second and third degree Diophantine equations, in spite of the many numerous results on this subject that have been gained by mathematicians with no little effort. The famous Thue theorem stating that the equations

$$a_0x^n + a_1x^{n-1}y + ... + a_{n-1}xy^{n-1} + a_ny^n = c$$

(a_i, rational integers, i=0,1,..,n;n>2)

has only a finite number of (rational) solutions is an amazing discovery. It leaves open the question how to find these solutions and what is their exact number, and one would conjecture that it will remain open for (all) times to come.

The questions considered in this part of the book are:

i) The equation, known (wrongly) as Pell's equation, namely

$$x^2 - my^2 = \pm 1,$$

ii) The equation $x^3 + my^3 + m^2z^3 - 3mxyz = 1$,

iii) and

$$\begin{vmatrix} x & y & z & u & v \\ mv & x & y & z & u \\ mu & mv & x & y & z \\ mz & mu & mv & x & y \\ my & mz & mu & mv & x \end{vmatrix} = 1.$$

Infinitely many solutions of each of these equations will be stated explicitly. Now, it is known that the solutions of EULER-PELL's equation is well exploited. Still, we found it necessary to include it here because of the simple method we shall use in solving this equation here, which has such a wide range of application in various branches of exact sciences. Also, we will derive some new combinatorial identities.

Since we are going to use some formulas obtained before introduce them here too.

(3.1) $$\left.\begin{array}{l} A_0^{(0)}=1,\ A_0^{(1)}=0,\ A_0^{(n+2)}=A_0^{(n)}+2DA_0^{(n+1)} \\ A_1^{(0)}=0,\ A_1^{(1)}=1,\ A_1^{(n+2)}=A_1^{(n)}+2DA_1^{(n+1)} \\ n=0,1,\ldots \end{array}\right\}$$

(3.2) $$A_1^{(v+1)}=A_0^{(v)}+2DA_0^{(v+1)}=A_1^{(v+2)}.$$

(3.3) $$A_0^{(2n)}=\sum_{i=0}^{n-1}\binom{2n-2-i}{i}(2D)^{2n-2-2i},\ n=1,2\ldots$$

for n⇒2n-1.

(3.4) $$A_0^{(2n+1)}=\sum_{i=0}^{n-1}\binom{2n-1-i}{i}(2D)^{2n-1-2i},\ n=1,2,\ldots$$

(3.5) $$e^f=A_0^{(fl)}+a_1^{(0)}{}_0^{(fl+1)}+\ldots+a_{n-1}^{(0)}A_0^{(fl+n-1)},\ f=1,2,\ldots$$

(3.6) $$\begin{vmatrix} A_0^{(v)}, & A_0^{(v+1)}, & \ldots. & A_0^{(v+n-1)} \\ A_1^{(v)}, & A_1^{(v+1)}, & \ldots. & A_1^{(v+n-1)} \\ \ldots. & \ldots. & \ldots. & \ldots. \\ A_{n-1}^{(v)}, & A_{n-1}^{(v+1)}, & \ldots. & A_{n-1}^{(v+n-1)} \end{vmatrix}=(-1)^{v(n-1)}$$

(3.7) $$\left.\begin{array}{l} A_0^{(0)}=1,\ A_0^{(1)}=A_0^{(2)}=0,\ A_0^{(n+3)}=A_0^{(n)}+3DA_0^{(n+1)}+3D^2A_0^{(n+2)} \\ n=0,1,\ldots \end{array}\right\}$$

(3.8) $$A_1^{(n+2)}=A_0^{(n+1)}+3DA_0^{(n+2)}$$

(3.9) $$A_2^{(n+2)}=A_0^{(n)}+3DA_0^{(n+1)}+3D^2A_0^{(n+2)}$$

(3.10) $$\prod_{i=1}^{k}a_{n-1}^{(i)}=A_0^{(k)}+a_1^{(k)}A_0^{(k+1)}+\ldots+a_{n-1}^{(k)}A_0^{(k+n-1)}$$

(3.11) $$\left.\begin{array}{l} A_0^{(n+3)}=\displaystyle\sum_{3y_1+2y_2+y_3=n}\binom{y_1+y_2+y_3}{y_1,\ \ y_2,\ \ y_3}3^{y_2+y_3}D^{y_2+2y_3} \\ n=0,1,\ldots;\dbinom{0}{0}\equiv 1 \end{array}\right\}$$

(3.12) $$\left.\begin{array}{l} A_0^{(0)}=1,A_0^{(1)}=A_0^{(2)}=A_0^{(3)}=A_0^{(4)}=0. \\ A_0^{(n+5)}=A_0^{(n)}+5DA_0^{(n+1)}+10D^2A_0^{(n+2)}+10D^3A_0^{(n+3)}+5D^4A_0^{(n+4)} \\ n=0,1,\ldots \end{array}\right\}$$

1*. **Euler-Pell's equation.**

We denote

(1*.1) $$\left.\begin{array}{l} w^2 = D^2 + 1 = m, \quad w = \sqrt{D^2 + 1}, \\ D \in N, \text{not a perfect square.} \end{array}\right\}$$

We obtain from (3.6) with n=2,

(1*.2) $$\left.\begin{array}{l} \begin{vmatrix} A_0^{(n)} & A_0^{(n+1)} \\ A_1^{(n)} & A_0^{(n+1)} \end{vmatrix} = (-1)^{(2-1)n} \\ n = 0,1,\ldots \end{array}\right\}$$

and from (3.1), (3.2)

$$\begin{vmatrix} A_0^{(n)} & A_0^{(n+1)} \\ A_0^{(n+1)} & A_0^{(n+2)} \end{vmatrix} = \begin{vmatrix} A_0^{(n)} & A_0^{(n+1)} \\ A_0^{(n+1)} & A_0^{(n)} + 2DA_0^{(n+1)} \end{vmatrix} =$$

$$A_0^{(n)^2} + 2DA_0^{(n)}A_0^{(n+1)} - A_0^{(n+1)^2} = (A_0^{(n)} + DA_0^{(n+1)})^2 - (D^2 + 1)A_0^{(n+1)^2},$$

(1*.3) $$(A_0^{(n)} + DA_0^{(n+1)})^2 - mA_0^{(n+1)^2}) = (-1)^n .$$

We have obtained EULER-PELL'S equations

(1*.4) $$\left.\begin{array}{l} x_n^2 - my_n^2 = (-1)^n, \quad n = 0,1,\ldots \\ x_n = A_0^{(n)} + DA_0^{(n+1)}, \quad y_n = A_0^{(n+1)} \end{array}\right\}$$

(1*.5) $$\left.\begin{array}{l} x_{2n} = A_0^{(2n)} + DA_0^{(2n+1)}, \quad y_{2n} = A_0^{(2n+1)} \\ x_{2n}^2 - my_{2n}^2 = 1. \end{array}\right\}$$

(1*.6) $$\left.\begin{array}{l} x_{2n+1} = A_0^{(2n+1)} + DA_0^{(2n+2)}, \quad y_{2n+1} = A_0^{(2n+2)} \\ x_{2n+1}^2 - my_{2n+1}^2 = -1. \end{array}\right\}$$

Thus we obtained infinitely many solutions of $x^2 - my^2 = \pm 1$, and, as is known from the theory of continued fractions, these are all solutions of these two equations, the so-called plus and minus cases of Euler-Pell's equations.

With (3.3), (3.4), formulas (1*.5), (1*.6) take the forms

$$x_{2n} = \sum_{i=0}^{n-1} \binom{2n-2-i}{i} (2D)^{2n-2-2i} + \frac{1}{2} \binom{2n-1-i}{i} (2D)^{2n-2i},$$

$$y_{2n} = \sum_{i=0}^{n-1} \binom{2n-1-i}{i} (2D)^{2n-1-2i},$$

$$x_{2n}^2 - my_{2n}^2 = 1, \ n = 0,1,...$$

$$x_{2n+1} = \sum_{i=0}^{n-1} \binom{2n-1-i}{i} (2D)^{2n-1-2i} + \sum_{i=0}^{n} \binom{2n-i}{i} (2D)^{2n+1-2i},$$

$$y_{2n+1} = \sum_{i=1}^{n} \binom{2n-i}{i} (2D)^{2n-2i}, \ n = 0,1,...$$

$$x_{2n+1}^2 - my_{2n+1}^2 = -1.$$

With the calculation of $A_0^{(v)}$ from the formulas introduced at the beginning we have:

$$x_0 = A_0^{(0)} + DA_0^{(1)} = A_0^{(0)} = 1; \ A_0^{(1)} = y_1 = 0,$$

$$x_0^2 = my_0^2 = 1^2 - (D^2+1)\cdot 0 = 1.$$

$$x_1 = A_0^{(1)} + DA_0^{(2)} = D, y_1 = A_0^{(2)} = 1,$$

$$x_1^2 - my_1^2 = D^2 - (D^2+1)\cdot 1 = -1.$$

$$x_2 = A_0^{(2)} + DA_0^{(3)} = 1 + 2D^2; \ y_2 = 2D,$$

$$x_2^2 - my_2^2 = (1 + 2D^2)^2 + (1 + D^2)\cdot 4D^2 = 1,$$

$$x_3 = A_0^{(3)} + DA_0^{(4)} = 3D + 4D^3; \ y_3 = 1 + 4D^2,$$

$$x_3^2 - my_3^2 = (3D + 4D^3)^2 - (1 + D^2)(1 + 4D^2)^2 = -1,$$

$$x_4 = 1 + 8D^2 + 8D^4; \ y_4 = 4D + 8D^3,$$

$$x_4^4 - my_4^2 = (1 + 8D^2 + 8D^4)^2 - (1 + D^2)(4D + 8D^3)^2 = 1.$$

2*. **Units in** Q(w), $w = \sqrt{D^2+1}$

It is clear that

(2*.1) $$e = w + D$$

is a unit in Q(w). For e is an integer, and e^{-1}=w-D, an integer.

The (BGEA) of $a^{(0)} = w + D$ is purely periodic with length of its primitive period m=1; hence we have from formula (3.5)

(2*.2) $$e^n = (w + D)^n = A_0^{(n)} + (w + D)A_0^{(n+1)}.$$

From (2*2) we get an interesting combinatorial identity

$$(w + D)^{2n} = A_0^{(2n)} + (w + D)A_0^{(2n+1)}$$

$$(w + D)^{2n} = A_0^{(2n)} + DA_0^{(2n+1)} + wA_0^{(2n+1)},$$

hence from (1*.5)

(2*.3) $$(w+D)^{2n} = x_{2n} + y_{2n} w.$$

With $w^2 = D^2 + 1 = m$, the reader will easily verify the formulas

(2*.4) $$(w+D)^{2n} = \left(\sum_{i=0}^{n} \binom{2n}{2i} D^{2i} m^{n-i}\right) + \left(\sum_{i=0}^{n-1} \binom{2n}{2i+1} D^{2i+1} m^{n-1-i}\right) w.$$

From (2*.3) and (2*.4), and using the expressions for x_{2n} and y_{2n} from the previous paragraph, we obtain the combinatorial identities

(2*.5) $$\left.\begin{aligned} &\sum_{i=0}^{n-1}\left[\binom{2n-2-i}{i}(2D)^{2n-2-2i} + \frac{1}{2}\binom{2n-1-i}{i}(2D)^{2n-2i}\right] \\ &\quad = \sum_{i=0}^{n} \binom{2n}{2i} D^{2i} (D^2+1)^{n-i}. \end{aligned}\right\}$$

(2*.6) $$\sum_{i=0}^{n-1}\left[\binom{2n-1-i}{i}(2D)^{2n-1-2i} = \sum_{i=0}^{n-1}\binom{2n}{2i+1} D^{2i+1} (D^2+1)^{n+1-i}\right].$$

Similar identities are obtainable from

$$(w+D)^{2n+1} = x_{2n+1} + y_{2n+1} w.$$

3*. **The cubic diophantine equations**.

We shall need formulas (3.6), (3.7), (3.8), (3.9) for n=3, viz.

(3*.1) $$\begin{vmatrix} A_0^{(n+1)} & A_0^{(n+2)} & A_0^{(n+3)} \\ A_1^{(n+1)} & A_1^{(n+2)} & A_0^{(n+3)} \\ A_2^{(n+1)} & A_2^{(n+2)} & A_2^{(n+3)} \end{vmatrix} = 1$$

(3*.2) $$\left.\begin{aligned} &A_0^{(0)} = 1,\ \ A_0^{(1)} = A_0^{(2)} = 0, \\ &A_0^{(n+3)} = A_0^{(n)} + 3DA_0^{(n+1)} + 3D^2 A_0^{(n+2)}, \\ &A_1^{(n+3)} = A_0^{(n+2)} + 3DA_0^{(n+3)} \\ &A_2^{(n+3)} = A_0^{(n+1)} + 3DA_0^{(n+2)} + 3D^2 A_0^{(n+3)}. \end{aligned}\right\}$$

Substituting in (3*.1) the values for $A_1^{(i)}, A_2^{(i)}$, i=n+3 from (3*.2), we obtain, after simple rearrangements

$$I=\begin{vmatrix} A_0^{(n+1)} & A_0^{(n+2)} & A_0^{(n+3)} \\ A_0^{(n)}+3DA_0^{(n+1)} & A_0^{(n+1)}+3DA_0^{(n+2)} & A_0^{(n+2)}+3DA_0^{(n+3)} \\ A_0^{(n+1)}+3DA_0^{(n)}+3D^2A_0^{(n+1)} & A_0^{(n)}+3DA_0^{(n+1)}+3D^2A_0^{(n+2)} & A_0^{(n+1)}+3DA_0^{(n+2)}+3D^2A_0^{(n+3)} \end{vmatrix}$$

$$=\begin{vmatrix} A_0^{(n+1)} & A_0^{(n+2)} & A_0^{(n+3)} \\ A_0^{(n)} & A_0^{(n+1)} & A_0^{(n+2)} \\ A_0^{(n-1)} & A_0^{(n)} & A_0^{(n+1)} \end{vmatrix}$$

We now denote

(3*.3) $$\left.\begin{aligned} & x=A_0^{(n-1)},\ \ y=A_0^{(n)},\ \ z=A_0^{(n+1)} \\ & n=1,2,\ldots \end{aligned}\right\}$$

and obtain for the above determinant

$$1=\begin{vmatrix} z & x+3Dy+3D^2z & y+3Dz+3D^2A_0^{(n+3)} \\ y & z & A_0^{(n+2)} \\ z & y & z \end{vmatrix}.$$

Subtracting from the first row the 3D multiple of the third and the $3D^2$ of the second, we obtain,

(3*.4) $$\begin{vmatrix} 2-3Dx-3D^2y & x & y \\ y & z & x+3Dy+3D^2z \\ x & y & z \end{vmatrix}=1.$$

We leave it to the reader to expand the determinant in (3*.4) to obtain the Diophantine equation of the third degree as

(3*.5) $$\left.\begin{aligned} & x^3+(9D^3+1)y^3+z^3+(9D^3-3)xyz+6Dx^2y+3D^2x^2z \\ & +12D^2y^2x+(9D^4-3D)y^2z-3Dz^2x-6D^2z^2y=1 \end{aligned}\right\}$$

Even for D=1, equation (3*.5) complicated form as

(3*.6) $x^3+10y^2+z^3+6xyz+6x^2y+3x^2z+12y^2x+6y^2z-3z^2x-6z^2y=1$

In section 1 we have calculated the solution triples $A_0^{(n)},\ A_0^{(n+1)}, A_0^{(n+2)}, n=0,1,\ldots$

(x,y,z)=(1,0,0), (0,0,1), (0,1,3), (1,3,12), (3,12,46), (12,46,177).

We shall check the solution

(x_3, y_3, z_3)=(1,3,12).

Substituting these values in (3*.6), we obtain:

1+270+1728+216+18+36+108+648-432-2592=1, 3025-3024=1

For larger values of D and n the verification of (3*.5) is only possible by computer, and without knowing (3*.3) even a computer would have its problems.

As we shall soon see, there is a much simpler third degree Diophantine equation which can be regarded as, and indeed in a certain case represents, a generalization of Euler-Pell's equation to the third degree.

4*. **Units in the cubic field**.

As we have seen in Chapter 2 the (BGEA) of the vector $a^{(0)} \in E_3$, with $w = \sqrt[3]{D^3 + 1}, D \in N, a^{(0)} = (w + 2D, w^2 + Dw + D^2)$, is purely periodic with length of primitive period m=1. Hence, by theorem 2 of Chapter 2 Section 6* and formula (3.10)

$$e = w^2 + Dw + D^2 \tag{4*.1}$$

is a unit in Q(w), and

$$\left.\begin{array}{c} e^v = A_0^{(v)} + (w + 2D)A_0^{(v+1)} + (w^2 + Dw + D^2)A_0^{(v+2)} \\ v = 0,1,... \end{array}\right\} \tag{4*.2}$$

Thus

$$\left.\begin{array}{r} (w^2 + Dw + D^2)^v = A_0^{(v)} + 2DA_0^{(v+1)} + D^2A_0^{(v+2)} \\ + (A_0^{(v+1)} + DA_0^{(v+2)}w + D^2A_0^{(v+2)}w^2 \end{array}\right\}. \tag{4*.3}$$

We shall find the field equation of the expressions (1*.3) in Q(w).

We denote

$$\left.\begin{array}{l} x_v = A_0^{(v)} + 2DA_0^{(v+1)} + D^2A_0^{(v+2)}, \\ y_v = A_0^{(v+1)} + DA_0^{(v+2)} \\ z_v = A_0^{(v+2)} \end{array}\right\} \tag{4*.4}$$

and have

$$\left.\begin{array}{l} e^v = x_v + y_v w + z_v w^2 \\ we^v = mz_v + x_v w + y_v w^2 \\ w^2e^v = my_v + mz_v w + x_v w^2 \\ m = w^3 = D^3 + 1 \end{array}\right\} \tag{4*.5}$$

Hence

(4*.6)
$$\begin{vmatrix} x_v & y_v & z_v \\ mz_v & x_v & y_v \\ my_v & mz_v & x_v \end{vmatrix} = 1$$

since N(e)=1, as the reader will easily verify.

Expanding the determinant in (4*.6), we obtain

(4*.7)
$$\left.\begin{array}{r} x_v^3 + my_v^3 + m^2 z_v^3 - 3mx_v y_v z_v = 1 \\ x_v, y_v, z_v \text{ from } (4*.4), v = 0,1,\ldots \end{array}\right\}$$

The Diophantine equation

$$x^3 + my^3 + m^2 z^3 - 3mxyz = 1$$

is indeed Euler-Pell's equation generalized to the third dimension. It is simpler compared with (3*.6) and it has as solutions (4*.4).

We shall verify formula (4*.5), first line for v=1,2. We have, from (4*.4).

$$x_1 = A_0^{(1)} + 2DA_0^{(2)} + D^2 A_0^{(3)} = D^2,$$

$$y_1 = A_0^{(2)} + DA_0^{(3)} = D,$$

$$z_1 = A_0^{(3)} = 1.$$

$$(D^2)^3 + (D^3+1)D^3 + (D^3+1)^2 \cdot 1 - 3(D^3+1)D^2 \cdot D =$$

$$D^6 + D^6 + D^3 + D^6 + 2D^3 + 1 - 3D^6 - 3D^3 = 1;$$

$$x_2 = A_0^{(2)} + 2DA_0^{(3)} + D^2 A_0^{(4)}$$

$$x_2 = 2D + D^2 \cdot 3D^2 = 2D + 3D^4,$$

$$y_2 = A_0^{(3)} + DA_0^{(4)} = 1 + 3D^3$$

$$z_2 = A_0^{(4)} = 3D^2.$$

We obtain substituting (x_2,y_2,z_2) in (4*.7)

$$1 + 18D^3 + 99D^2 + 162D^9 + 81D^{12} - 3(6D^3 + 33D^6 + 54D^9 + 27D^{12}) = 1,$$

We shall now extract a few interesting identities from formula (4*.3).

We have, by the binomial theorem

$$(w^2 + Dw + D^2)^{3n} = \sum_{i=0}^{3n} (w^2)^{2n-i}(Dw + D^2)^i =$$

$$= \sum_{i=0}^{3n} \binom{3n}{i} w^{6n-2i} \sum_{j=0}^{i} \binom{i}{j} (Dw)^{i-j} D^{2j} =$$

$$= \sum_{\substack{i=0\\ j=0,1,\ldots,i}}^{3n} \binom{3n}{i} w^{6n-2i} \binom{i}{j} D^{i+j} w^{i-j} =$$

$$= \sum_{\substack{i=0\\ j=0,1,\ldots,i}}^{3n} \binom{3n}{i}\binom{i}{j} w^{6n-(i+j)} D^{i+j}.$$

In the sum $\sum \binom{3n}{i}\binom{i}{j} w^{6n-(i+j)} D^{i+j}$, i=0,1,..,3n; j=0,1,..,i; we want to find the coefficient of powers of w^{3n}, so that since $w^3 = m = D^3 + 1$, this sum becomes rational. For this purpose we have to set $i+j \equiv 0(3)$ and obtain

(4*.8) $$(w^2+Dw+D^2)^{3n} = \sum_{\substack{i+j=3s\le 6n\\ s=0,1,\ldots,2n\\ 0\le j\le i\le i\le 3n}} \binom{3n}{i}\binom{i}{j} w^{6n-3s} D^{3s}$$

(4*.8) is an appealing formula for the expression $(w^2 + Dw + D^2)^{3n}$, through this expression could also be calculated by the multinomial theorem.
We have, in order to illustrate its application; n=1, s=0,1,2.

$$s = 0; i = j = 0;$$
$$s = 1; i = 2, j = 1; i = 3; j = 0;$$
$$s = 2; i = 3, j = 3. (i \le 3).$$

Hence we have for the rational part of e^3:

$$(w^2 + Dw + D^2)^3 = \binom{3}{0}\binom{0}{0} w^6 + \binom{3}{2}\binom{2}{1} w^3 D^3 + \binom{3}{3}\binom{3}{0} w^3 D^3 +$$
$$+ \binom{3}{3}\binom{3}{3} D^6 = w^6 + 7D^3 w^3 + D^6,$$

$$[w^2 + (Dw + D^2)]^3 = w^6 + 3w^4(Dw + D^2) + 3w^2(Dw + D^2)^2 + (Dw + D^2)^3 =$$
$$= w^6 + 3Dw^5 + 3D^2w^4 + 3D^2w^4 + 6D^3w^3 +$$
$$+ 3w^2D^4 + D^3w^3 + 3D^4w^2 + 3D^5w + D^6.$$

The rational members of this sum are $w^6 + 6D^3w^3 + D^3w^3 + D^6$, as was calculated above, with $w^3 = m = D^3 + 1$. The formula (4*.8) is easily applicable since there is no difficulty to solve the linear equations i+j=3s.

We shall still find the rational part of $e^6 = (w^2 + Dw + D^2)^6$. By formula (4*.8), with

$$\begin{aligned} &n = 2,\ s = 0,1,2,3,4;\ i + j = 3s,\quad j \le i \le 6;\\ &s = 0;\ \ i = j = 0;\\ &s = 1;\ \ i = 3,\ j = 0,\\ &\qquad\ \ i = 2,\ j = 1;\\ &s = 2;\ \ i = 6,\ j = 0,\\ &\qquad\ \ i = 5,\ j = 1;\\ &\qquad\ \ i = 4,\ j = 2;\\ &\qquad\ \ i = 3,\ j = 3;\\ &s = 3;\ \ i = 6,\ j = 3,\\ &\qquad\ \ i = 5,\ j = 4;\\ &s = 4;\ \ i = 6,\ j = 6; \end{aligned}$$

we obtain

$$\begin{aligned} &\binom{6}{0}\binom{0}{0}w^{12} + \left[\binom{6}{3}\binom{3}{0} + \binom{6}{2}\binom{2}{1}\right]w^9D^3 +\\ &+\left[\binom{6}{6}\binom{6}{0} + \binom{6}{5}\binom{5}{1} + \binom{6}{4}\binom{4}{2} + \binom{6}{3}\binom{3}{3}\right]w^6D^6 +\\ &+\left[\binom{6}{6}\binom{6}{3} + \binom{6}{5}\binom{5}{4}w^3D^9\right] + \binom{6}{6}\binom{6}{6}D^{12} =\\ &= m^{12} + 50w^9D^3 + 141w^6D^6 + 50w^3D^9 + D^{12} =\\ &= m^4 + 50m^3D^3 + 141m^2D^6 + 50mD^9 + D^{12},\\ &m = D^3 + 1 = w^3. \end{aligned}$$

We thus have the final results, viz. The rational part of $e^{3n} = (w^2 + Dw + D^2)^{3n}$ equals

(4*.9) $$\sum_{\substack{i+j=3s\le 6n\\ s=0,1,\ldots,2n\\ 0\le j\le i\le i\le 3n}} \binom{3n}{i}\binom{i}{j} m^{2n-s}D^{3s},\ m = D^3 + 1.$$

We shall now find the coefficient of w in

$$\sum_{\substack{i=0 \\ j=0,1,\ldots,i}}^{3n} \binom{3n}{i}\binom{i}{j} w^{6n-(i+j)} D^{i+j}$$

and demand, to this end,

$$\left.\begin{aligned} & 6n-(i+j) \equiv 1 \pmod 3, \\ & i+j \equiv 2(3), i+j = 3s+2 \\ & s = 0,1,\ldots,2n-1 \end{aligned}\right\} \tag{4*.10}$$

and obtain thus, that this coefficient equals

$$\sum_{\substack{i+j=3s+2 \\ s=0,1,\ldots,2n-1 \\ 0\le j\le i\le 3n}} \binom{3n}{i}\binom{i}{j} w^{6n-(3s+2)} D^{3s+2}. \tag{4*.11}$$

But

$$\begin{aligned} w^{6n-(3s+2)} &= w^{6n-(3s+3)+1} = \\ &= w^{3[2n-(s+1)]+1} = m^{2n-(s+1)} w, \\ & m = w^3 = D^3 + 1. \end{aligned}$$

Hence,

The coefficient of w in $(w^2 + Dw + D^2)^{3n}$ equals

$$\sum_{\substack{i+j=3s+2 \\ s=0,1,\ldots,2n-1 \\ 0\le j\le i\le 3n}} D^{3s+2} m^{2n-s-1} \binom{3n}{i}\binom{i}{j} \tag{4*.12}$$

Illustration of (4*.12):

$$n = 1; s = 0,1; s = 0; i = 2, j = 0; i = 1, j = 1;$$
$$s = 1; i = 3, j = 2$$

The coefficient of w in the expansion of $(w^2 + Dw + D^2)^3$ equals

$$D^2 m\left[\binom{3}{2}\binom{2}{0} + \binom{3}{1}\binom{1}{1}\right] + D^5\left[\binom{3}{3}\binom{3}{2}\right] = 6D^2 + 3D^5,$$

as the reader can verify.

$$n = 2; s = 0,1,2,3;\ 3n = 6 \geq i.$$
$$s = 0; i = 2; j = 0; i = 1; j = 1;$$
$$s = 1; i = 5, j = 0; i = 4; j = 1; i = 3, j = 2;$$
$$s = 2; i = 6; j = 1; i = 5, j = 2; i = 4, j = 3;$$
$$s = 3; i = 6, j = 5.$$

The coefficient of w in $(w^2 + Dw + D^2)^6$ equals

$$\left[\binom{6}{2}\binom{2}{0} + \binom{6}{1}\binom{1}{1}\right]D^2m^3 + \left[\binom{6}{5}\binom{5}{0} + \binom{6}{1}\binom{1}{1}\right]D^5m^2 +$$
$$+\left[\binom{6}{6}\binom{6}{1} + \binom{6}{5}\binom{5}{2} + \binom{6}{4}\binom{4}{3}\right]D^8m + \left[\binom{6}{0}\binom{6}{5}\right]D^{11} =$$
$$= 21D^2m^3 + 12D^5m^2 + 126D^8m + 6D^{11}.$$

The reader will now prove without any difficulty that:

The coefficient of w^2 in $(w^2 + Dw + D^2)^{3n}$ equals

$$\sum_{\substack{i+j=3s+1\\ s=0,1,\ldots,2n-1,\\ 0\leq j\leq i\leq 3n}} \binom{3n}{i}\binom{i}{j} D^{3s+1} m^{2n-s-1} \tag{4*.13}$$

But by (4*.3) we have

$$(w^2 + Dw + D^2)^{3n} = [A_0^{(3n)} + 2DA_0^{(3n+1)} + D^2A_0^{(3n+2)}] +$$
$$+ [A_0^{(3n+1)} + DA_0^{(3n+2)}]w + D^2A_0^{(3n+2)}w^2.$$

With (4*.9), (4*.12), (4*.13) we obtain the identities

$$\sum_{\substack{i+j=3s\leq 6n\\ 0\leq j\leq i\leq 3n}} \binom{3n}{i}\binom{i}{j} m^{2n-2} D^{3s} = A_0^{(3n)} + 2DA_0^{(3n+1)} + D^2A_0^{(3n+2)}. \tag{4*.14}$$

$$\sum_{\substack{i+j=3s+2\leq 6n-1\\ 0\leq j\leq i\leq 3n}} \binom{3n}{i}\binom{i}{j} D^{3s+2} m^{2n-s-1} = A_0^{(3n+1)} + DA_0^{(3n+2)}. \tag{4*.14a}$$

$$\sum_{\substack{i+j=3s+1\leq 6n-2\\ 0\leq j\leq i\leq 3n}} \binom{3n}{i}\binom{i}{j} D^{3s+1} m^{2n-s-1} = D^2A_0^{(3n+2)}. \tag{4*.14b}$$

If we substitute in (4*.14),(4*.14a),(4*.14b) the values of $A_0^{(3n)}, A_0^{(3n+1)}, A_0^{(3n+2)}$, we indeed arrive at some new combinatorial identities. We proceed to obtain further identities for the third dimension.

5*. **More identities**.

We return to formula (4*.2)

$$(w^2+Dw+D^2)^v = A_0^{(v)} + (w+2D)A_0^{(v+1)} + (w^2+Dw+D^2)A_0^{(v+2)},$$

and have with $(w-D)(w^2+Dw+D^2)=1$,

$$(w-D)^v = \frac{1}{A_0^{(v)} + (w+2D)A_0^{(v+1)} + (w^2+Dw+D^2)A_0^{(v+2)}}. \quad (5^*.1)$$

We want to rationalize the denominator in (5*.1) so that

$$[A_0^{(v)} + (w+2D)A_0^{(v+1)} + (w^2+Dw+D^2)A_0^{(v+2)}](a+bw+cw^2)=1. \quad (5^*.2)$$

We obtain, with a, b, c rationals,

$$\left.\begin{aligned}&(A_0^{(v)}+2DA_0^{(v+1)}+D^2A_0^{(v+2)})a+mA_0^{(v+2)}b+(A_0^{(v+1)}+DA_0^{(v+2)})mc=1,\\&(A_0^{(v+1)}+DA_0^{(v+2)})a+(A_0^{(v)}+2DA_0^{(v+1)}+D^2A_0^{(v+2)})b+mA_0^{(v+2)}c=0,\\&A_0^{(v+2)}a+(A_0^{(v+1)}+DA_0^{(v+2)})b+(A_0^{(v)}+2DA_0^{(v+1)}+D^2A_0^{(v+2)})c=0\end{aligned}\right\} \quad (5^*.3)$$

The determinant of this system of equations (5*.3) equals, with

$$x_v = A_0^{(v)} + 2DA_0^{(v+1)} + D^2A_0^{(v+2)};\ y_v = A_0^{(v+1)} + DA_0^{(v+2)};\ z_v = A_0^{(v+2)}$$

$$\begin{vmatrix} x_v & mz_v & my_v \\ y_v & x_v & mz_v \\ z_v & y_v & x_v \end{vmatrix} = x_v^3 + my_v^3 + m^2z_v^3 - 3mx_vy_vz_v = 1.$$

Hence we obtain from (5*.3)

$$a = \begin{vmatrix} 1 & mz_v & my_v \\ 0 & x_v & mz_v \\ 0 & y_v & x_v \end{vmatrix} = x_v^2 - my_vz_v$$

$$b = \begin{vmatrix} x_v & 1 & my_v \\ y_v & 0 & mz_v \\ z_v & 0 & x_v \end{vmatrix} = mz_v^2 - x_vy_v$$

$$c = \begin{vmatrix} x_v & mz_v & 1 \\ y_v & x_v & 0 \\ z_v & y_v & 0 \end{vmatrix} = y_v^2 - x_vz_v.$$

Thus we have obtained the identity.

$$(w-D)^2 = x_v^2 - my_vz_v + (mz_v^2 - x_vy_v)w + (y_v^2 - x_vz_v)w^2 \quad \text{or}$$

$$(w-D)^{3v} = x_{3v}^2 - my_{3v}z_{3v} + (mz_{3v}^2 - 3x_{3v}y_{3v})w + (y_{3v}^2 - 3x_{3v}z_{3v})w^2. \quad (5^*.4)$$

Expanding $(w-D)^{3v}$, we obtain, with $w^3 = m = (D^3+1)$

$$(5^*.5)\quad \left.\begin{aligned}(w-D)^{3v} &= \sum_{i=0}^{v}(-1)^i\binom{3v}{3i}m^{v-i}D^{3i} + \left(\sum_{i=0}^{v-1}(-1)^i\binom{3v}{3i+2}m^{v-1-i}D^{3i+2}\right)w + \\ &+\left(\sum_{i=0}^{v-1}(-1)^{i+1}\binom{3v}{3i+1}m^{v-1-i}D^{3i+1}\right)w^2\end{aligned}\right\}$$

With (5*.4), (5*.5) we obtain some new identities

$$(5^*.6)\quad \left.\begin{aligned}&\sum_{i=0}^{v}(-1)^i\binom{3v}{3i}m^{v-i}D^{3i} = x_{3v}^2 - my_{3v}z_{3v};\\ &\sum_{i=0}^{v-1}(-1)^i\binom{3v}{3i+2}m^{v-1-i}D^{3i+2} = mz_{3v}^2 - x_{3v}y_{3v};\\ &\sum_{i=0}^{v-1}(-1)^{i+1}\binom{3v}{3i+1}m^{v-1-i}D^{3i+1} = y_{3v}^2 - x_{3v}z_{3v};\\ &\qquad v=0,1,\ldots;\ x_v, y_v, z_v \text{ from}(4^*.4).\end{aligned}\right\}$$

Substituting for x_v, y_v, z_v, the values from (4*.4), and the values of $A^{(v)}$ from (3.11) the identities (5*.6) take the form

6*. **Fifth degree diophantine equations**.

We return to formula (2*.6) with n=5, and obtain

$$(6^*.1)\quad \begin{vmatrix} A_0^{(n+4)} & A_0^{(n+5)} & A_0^{(n+6)} & A_0^{(n+7)} & A_0^{(n+8)} \\ A_1^{(n+4)} & A_1^{(n+5)} & A_1^{(n+6)} & A_1^{(n+7)} & A_1^{(n+7)} \\ A_2^{(n+4)} & A_2^{(n+5)} & A_2^{(n+6)} & A_2^{(n+7)} & A_2^{(n+7)} \\ A_3^{(n+4)} & A_3^{(n+5)} & A_3^{(n+6)} & A_3^{(n+7)} & A_3^{(n+7)} \\ A_4^{(n+4)} & A_4^{(n+5)} & A_4^{(n+6)} & A_4^{(n+7)} & A_4^{(n+7)} \end{vmatrix} = (-1)^{(5-1)(n+4)} = 1,$$

$$n=0,1,\ldots$$

Substituting for $A_i^{(v)}$, i=1,2,3,4; v=n+4,…,n+8; their representation as forms of $A_0^{(n+j)}$, j=1,2,3,4, we obtain the matrix equality.

$$(6^*.2)\quad \begin{vmatrix} A_0^{(n+4)} & A_0^{(n+5)} & A_0^{(n+6)} & A_0^{(n+7)} & A_0^{(n+8)} \\ A_0^{(n+3)} & A_0^{(n+4)} & A_0^{(n+5)} & A_0^{(n+6)} & A_0^{(n+7)} \\ A_0^{(n+2)} & A_0^{(n+3)} & A_0^{(n+4)} & A_0^{(n+5)} & A_0^{(n+6)} \\ A_0^{(n+1)} & A_0^{(n+2)} & A_0^{(n+3)} & A_0^{(n+4)} & A_0^{(n+5)} \\ A_0^{(n)} & A_0^{(n+1)} & A_0^{(n+2)} & A_0^{(n+3)} & A_0^{(n+4)} \end{vmatrix} = 1,$$

We denote

(6*.3) $$A_0^{(n+4)} = v, A_0^{(n+3)} = u, A_0^{(n+2)} = z, A_0^{(n+1)} = y, A_0^{(n)} = x$$

and with formula (3.12), viz.

$$A_0^{(n+5)} = A_0^{(n)} + 5D\,A_0^{(n+1)} + 10D^2\,A_0^{(n+2)} + 10D^3\,A_0^{(n+3)} + 5D^4\,A_0^{(n+4)}.$$

We also denote

(6*.4) $$5D = a_4, 10D^2 = a_3, 10D^3 = a_2, 5D^4 = a_1,$$
$$(a_4 = b_1^{(0)}; a_3 = b_2^{(0)}; a_2 = b_3^{(0)}, a_1 = b_4^{(0)}).$$

We then proceed as follows (in order to represent (6*.2) as an expression in powers of x,y,z,u,v):

i) from the first row we subtract the a_1 multiple of the second row, then the a_2 multiple of the third row, then the a_3 multiple of the fourth row, then the a_4 multiple of fifth row.

ii) From the second row we subtract the a_1 multiple of third row, then the a_2 multiple of the fourth row, then the a_3 multiple of the fifth row.

iii) From the third row we subtract the a_1 multiple of fourth row, then the a_2 multiple of the fifth row.

iv) From the fourth row we subtract the a_1 multiple of the fifth row, and obtain, always applying formula (3.12) and the notations (6*.3), (6*.4

(6*.5)

$$\begin{vmatrix} v - a_1u - a_2z - a_3y - a_4x & x & y & z & u \\ u - a_1z - a_2y - a_3x & v - a_1u - a_2z - a_3y & x + a_4y & y + a_4z & z + a_4u \\ z - a_1y - a_2x & u - a_1z - a_2y & v - a_1u - a_2z & x - a_4y + a_3z & y - a_4z + a_3u \\ y - a_1x & z - a_1y & u - a_1z & v - a_1u & x + a_4y + a_3z + a_2u \\ x & y & z & u & v \end{vmatrix} = 1$$

with the values of a_1, a_2, a_3, a_4 from (6*.4), x,y,z,u,v from (6*.3) where n=0,1,…The expansion of the determinant (6*.5) would yield the expression. Even with D=1, it will still be very complicated. For n=0, x=1, y = z = u = v = 0, the determinant in (6*.5) becomes

$$\begin{vmatrix} -a_4 & 1 & 0 & 0 & 0 \\ -a_3 & 0 & 1 & 0 & 0 \\ -a_2 & 0 & 0 & 1 & 0 \\ -a_1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = 1,$$

and for n = 1, u = z = y = x = 0, v = 1, the determinant becomes

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1,$$

but these elementary determinants can hardly serve as a verification for formula (6*.5). For n=2 the test is also simple.

Let try for n=3,

$$(A_0^{(3)}, A_0^{(4)}, A_0^{(5)}, A_0^{(6)}, A_0^{(7)}) = (0,0,1,a_1,a_2+a_1^2) = (x,y,z,u,v)$$

$$\begin{vmatrix} a_2+a_1^2-a_1^2-a_2 & 0 & 0 & 1 & a_1 \\ a_2-a_1 & a_2+a_1^2-a_1^2-a_2 & 0 & a_4 & 1+a_1a_4 \\ 1 & a_1-a_1 & a_2+a_1^2-a_1^2-a_2 & a_3 & a_4+a_1a_3 \\ 0 & 1 & a_1-a_1 & a_2+a_1^2-a_1^2 & a_3+a_1a_2 \\ 0 & 0 & 1 & a_1 & a_2+a_1^2 \end{vmatrix} =$$

$$\begin{vmatrix} 0 & 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & a_4 & 1+a_1a_4 \\ 1 & 0 & 0 & a_3 & a_4+a_1a_3 \\ 0 & 1 & 0 & a_2 & a_3+a_1a_2 \\ 0 & 0 & 1 & a_1 & a_2+a_1^2 \end{vmatrix} =$$

and subtracting the a_1 multiple of the fourth column from the fifth

$$= \begin{vmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_4 & 1 \\ 1 & 0 & 0 & a_3 & a_4 \\ 0 & 1 & 0 & a_2 & a_3 \\ 0 & 0 & 1 & a_1 & a_2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & a_4 & 1 \\ 1 & 0 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & a_3 \\ 0 & 1 & a_2 \end{vmatrix} = 1.$$

7*. **Fifth degree identities**.

As we have seen, the (BGEA) of the fixed vector

$a^{(0)} =$

$(w+4D, w^2+3Dw+6D^2, w^3+2Dw^2+3D^2w+4D^3, w^4+Dw^3+D^2w^2+D^3w+D^4)$

is purely periodic with length of the primitive period *l*=1. Hence we have the formula

$$
\begin{aligned}
e^{n} &= (w^{4}+Dw^{3}+D^{2}w^{2}+D^{3}w+D^{4})^{n} = \\
&A_{0}^{(n)} + (w+4D)A_{0}^{(n+1)} + (w^{2}+3Dw+6D^{2})A_{0}^{(n+2)} + \\
&+(w^{3}+2Dw^{2}+3D^{2}w+4D^{3})A_{0}^{(n+3)} + \\
&+(w^{4}+Dw^{3}+D^{2}w^{2}+D^{3}w+D^{4})A_{0}^{(n+4)}, \\
&\quad n=0,1,2,... \\
&A_{0}^{(v)}\ (v=5,6,...)\ \text{from CH.2.}
\end{aligned}
\tag{7*.1}
$$

From (7*.1) we obtain

$$
\begin{aligned}
&(w^{4}+Dw^{3}+D^{2}w^{2}+D^{3}w+D^{4})^{5n} = \\
&= A_{0}^{(5n)} + 4DA_{0}^{(5n+1)} + 6D^{2}A_{0}^{(5n+2)} + 4D^{3}A_{0}^{(5n+3)} + D^{4}A_{0}^{(5n+4)} + \\
&+(A_{0}^{(5n+1)} + 3DA_{0}^{(5n+2)} + 3D^{2}A_{0}^{(5n+3)} + D^{3}A_{0}^{(5n+4)})w + \\
&+(A_{0}^{(5n+2)} + 2DA_{0}^{(5n+3)} + D^{2}A_{0}^{(5n+4)})w^{2} + \\
&+(A_{0}^{(5n+3)} + DA_{0}^{(5n+4)})w^{3} + A_{0}^{(5n+4)})w^{4}.
\end{aligned}
\tag{7*.2}
$$

We shall now arrange $(w^{4}+Dw^{3}+D^{2}w^{2}+D^{3}w+D^{4})^{5n}$ in descending powers of w. The first step will be to achieve this arrangement in powers of w^{5s}, s=0, 1, 2, 3, …,4n, since the highest power of w in that expression is w^{20n}, so we look for the rational part of it. We have by the multinomial theorem

$$
(w^{4}+Dw^{3}+D^{2}w^{2}+D^{3}w+D^{4})^{5n} =
$$

$$
\sum_{\substack{4y_1+3y_2+2y_3+y_4=k \\ y_2+2y_3+3y_4+4y_5=20n-k, \\ k=0,1,...,20n}} \binom{y_1+y_2+y_3+y_4+y_5}{y_1, y_2, y_3, y_4, y_5} w^{4y_1+3y_2+2y_3+y_4} \cdot D^{y_2+2y_3+3y_4+4y_5}
\tag{7*.3}
$$

since the sum of the exponents of w and D in the above expansion equals 20n=k+(20n-k). We also have from (7*.3)

$$
\begin{aligned}
&4y_1+4y_2+4y_3+4y_4+4y_5=20n \\
&y_1+y_2+y_3+y_4+y_5=5n
\end{aligned}
\tag{7*.4}
$$

Since we are looking for 5-multiples of the exponents of w-hence also of D-we obtain from (7*.3), (7*.4):

The rational part in the expansion of

$$(w^4 + Dw^3 + D^2w^2 + D^3w + D^4)^{5n} \text{ equals}$$

(7*.5)
$$\sum_{\sum_{i=1}^{4}(5-i)y_1 = 5s \le 20n} \binom{5n}{y_1, y_2, y_3, y_4, y_5} m^s D^{20n-5s}$$

$$s \ge 0, m = w^5 = (D^5 + 1).$$

The equation follows from $y_1 + y_2 + y_3 + y_4 + y_5 = 5n$ in the multinomial coefficient.

As an illustration to (7*.5) we shall find the rational part in the expansion of $(w^4 + Dw^3 + D^2w^2 + D^3w + D^4)^5, n = 1.$ We obtain from (7*.5) that this equals

(7*.6)
$$\sum_{\sum_{i=1}^{4}(5-i)y_1 = 5s \le 20n} \binom{5}{y_1, y_2, y_3, y_4, y_5} m^s D^{20-5s}$$

We solve the equations, s=0, 1, 2, 3, 4

$$s = 0; 4y_1 + 3y_2 + 2y_3 + y_4 = 0, \ y_1 + y_2 + y_3 + y_4 + y_5 = 5$$

$$y_1 = y_2 = y_3 = y_4 = 0, y_5 = 5.$$

The corresponding member in (7*.6) equals

$$\binom{5}{5} m^0 D^{20-0} = \underline{D^{20}}.$$

$$s = 1; 4y_1 + 3y_2 + 2y_3 + y_4 = 5, y_1 + y_2 + y_3 + y_4 + y_5 = 5,$$

$$y_1 = y_4 = 1; y_2 = y_3 = 0; y_5 = 3.$$

$$y_1 = 0; \ y_2 = y_3 = 1; y_4 = 0; y_5 = 3.$$

$$y_1 = 0; y_2 = 1; y_3 = 0; y_4 = 2; y_5 = 2.$$

$$y_1 = y_2 = 0; y_3 = 1; y_4 = 3; y_5 = 1.$$

$$y_1 = y_2 = 0; y_3 = 2; y_4 = 1; y_5 = 2.$$

$$y_1 = y_2 = y_3 = y_5 = 0; y_4 = 5.$$

The corresponding member in (7*.6) equals

$$\left[\binom{5}{1,1,0,0,3} + \binom{5}{0,1,1,0,3} + \binom{5}{0,1,0,2,2} + \binom{5}{0,0,1,3,1} + \right.$$
$$\left. + \binom{5}{0,0,2,1,2} + \binom{5}{0,0,0,0,5}\right] mD^{15} = \underline{121m^{15}}.$$

$$s = 2; 4y_1 + 3y_2 + 2y_3 + y_4 = 10; y_1 + y_2 + y_3 + y_4 + y_5 = 5.$$

We shall write $(y_1, y_2, y_3, y_4, y_5)$ for the solution of the above linear equations.

$$(2,0,1,0,2);\ (2,0,0,2,1);\ (1,2,0,0,2);$$
$$(1,1,1,1,1);\ (0,3,0,1,1);\ (0,2,2,0,1);$$
$$(0,1,3,1,0);\ (0,2,1,2,0);\ (0,1,3,1,0);$$
$$(0,0,5,0,0);\ (1,0,3,0,1);\ (1,0,2,2,0).$$

The corresponding member in (7*.6) equals

$$\left[\binom{5}{2,0,1,0,2}+\binom{5}{2,0,0,2,1}+\binom{5}{1,2,0,0,2}+\binom{5}{1,1,1,1,1,}+\right.$$
$$+\binom{5}{0,3,0,1,1}+\binom{5}{0,2,2,0,1}+\binom{5}{0,1,3,1,0}+\binom{5}{0,2,1,2,0}+$$
$$\left.+\binom{5}{0,1,3,1,0}+\binom{5}{0,0,5,0,0}+\binom{5}{1,0,3,0,1}+\binom{5}{1,0,2,2,0}\right]=$$
$$=(30+30+30+120+20+30+20+30+20+1+20+30)m^2D^{10}=\underline{381m^2D^{10}}.$$

$$s=3; 4y_1+3y_2+2y_3+y_4=15;\ y_1+y_2+y_3+y_4+y_5=5;$$
$$(3,1,0,0,1);\ (2,2,0,1,0);\ (3,0,1,1,0);$$
$$(2,1,2,0,0);\ (1,3,1,0,0,);\ (0,5,0,0,0);$$

The corresponding members in (7*.6) equals

$$\left[\binom{5}{3,1,0,0,1}+\binom{5}{2,2,0,1,0}+\binom{5}{3,0,1,1,0}+\binom{5}{2,1,2,0,0}+\right.$$
$$\left.+\binom{5}{1,3,1,0,0}+\binom{5}{0,5,0,0,0}\right]m^3D^5=$$
$$=(20+30+20+30+20+1)m^3D^5=\underline{121m^3D^5}$$

$$s=4; 4y_1+3y_2+2y_3+y_4=20;\ y_1+y_2+y_3+y_4+y_5=5.$$

The only solution is (5,0,0,0,0) and the corresponding member in (7*.6) equals

$$\binom{5}{5,0,0,0,0}m^4=\underline{m^4}.$$

Thus the formula (7*.5) yields, for n=1, the sum

(7*.7) $$m^4+121m^3D^5+381m^2D^{10}+121mD^{15}+D^{20}.$$

From the other side we have

$$(w^4 + Dw^3 + D^2w^2 + D^3w + D^4)^5 =$$
$$= w^{20} + 5w^{19}D + 15w^{18}D^2 + 35w^{17}D^3 + 70w^{16}D^4 +$$
$$+ 121w^{15}D^5 + 185w^{14}D^6 + 255w^{13}D^7 + 320w^{12}D^8 +$$
$$(7^*.8) \quad + 365w^{11}D^9 + 381D^{10}w^{10} + 365D^{10}w^{10} + 365w^9D^{11} + 320w^8D^{12} +$$
$$+ 255w^7D^{13} + 185w^6D^{14} + 121w^5D^{15} + 70w^4D^{16} +$$
$$+ 35w^3D^{17} + 15w^2D^{18} + 5wD^{19} + D^{20}.$$

That the expansion in (7*.8) is symmetric (the coefficients of w^iD^{20-i} and $w^{20-i}D^i$, are equal) is clear. The rational part equals

$$w^{20} + 121w^{15}D^5 + 381w^{10}D^{10} + 121w^5D^{15} + D^{20} =$$
$$= m^4 + 121m^3D^5 + 381m^2D^2 + 121mD^{15} + D^{20}$$

as should be by (7*.7).

Comparing formulas (7*.2) with (7*.5), we obtain the identity

$$(7^*.9) \quad \sum_{0 \le \sum_{i=1}^{4}(5-i)y_i = 5s \le 20n} \binom{5n}{y_1, y_2, y_3, y_4, y_5}^{m^sD^{20n-5s}} = \sum_{i=0}^{4}\binom{4}{i} D^i A_0^{(5n+i)}.$$

$$n = 1,2,...; A_0^{(v)} \text{ form} (7^*.6), v = 5,6,...$$

Substitution of the values of $A_0^{(v)}$ from (7*.6) in (7*.9) would yield a new expression for (7*.9). The reader can prove the statements:

The coefficient of w in the expansion of

$$(w^4 + Dw^3 + D^2w^2 + D^3w + D^4)^{5n} \text{ equals}$$

$$(7^*.10) \quad \sum_{\sum_{i=1}^{4}(5-i)y_i = 5s+1 \le 20n;} \binom{5n}{y_1, y_2, y_3, y_4, y_5} m^sD^{20n-5s-1}$$

$$s = 0,1,...,4n-1$$

Furthermore, the coefficient of w^i in the expansion of

$$(w^4 + Dw^3 + D^2w^2 + D^3w + D^4)^{5n} \text{ equal, with } i=0,1,2,3,4.$$

$$(7^*.11) \quad \sum_{i \le \sum_{i=1}^{4}(5-i)y_i = 5s+i \le 20n;} \binom{5n}{y_1, y_2, y_3, y_4, y_5} m^sD^{20n-5s-i}$$

$$s = 0,1,...,4n-1, \ i = 0,1,...,4$$

Comparing (7*.2) with (7*.11) we have finally the five identities,

$$\sum_{i\le\sum_{ji=1}^{4}(5-j)y_j=5s+i\le 20n;}\binom{5n}{y_1,y_2,y_3,y_4,y_5}m^sD^{20n-5s-i}=$$

(7*.12)
$$=\sum_{j=0}^{4-i}\binom{4-i}{j}D^jA_0^{(5n+i+j)}$$
$$i=0,1,2,3,4.$$

We shall give a verification for formula (7*.12) with i=0, formula (7*.9), D=1, n=1;we have $m=D^3+1=2, A_0^{(0)}=1, A_0^{(1)}=A_0^{(2)}=A_0^{(3)}=A_0^{(4)}=0,$
$A_0^{(n+5)}=A_0^{(n)}+5A_0^{(n+1)}+10A_0^{(n+3)}+10A_0^{(n+3)}+5A_0^{(n+4)}, A_0^{(5)}=1, A_0^{(6)}=5,$
$A_0^{(7)}=35, A_0^{(8)}=235, A_0^{(9)}=1580.$

This yields

$$16+121\cdot 8+381\cdot 4+121\cdot 2+1=1+20+210+940+1580=2751.$$

It is also easy to verify the identities (7*.12) for n=2.

8*. **More about units and identities**.

Since $w^5-D^5=(w-D)(w^4+Dw^3+D^2w^2+D^3w+D^4)=1$, we have also, $(w^5=D^5+1)$

$$e^{-v}=(w-D)^v=\frac{1}{(w^4+Dw^3+D^2w^2+D^3w+D^4)^4},$$

and with formula (7*.2), and setting v=5n,

(8*.1)
$$(w-D)^{5n}=\frac{1}{a_5+a_4w+a_3w^2+a_2w^3+a_1w^4}$$
$$a_{5-i}=\sum_{j=0}^{4-i}\binom{4-i}{j}D^jA_0^{(5n+i+j)},\ \ i=0,...,4\ \ (\text{from}\,(7*.12)).$$

We shall now rationalize the denominator in (8*.1) and demand

(8*.2) $1=(a_5+a_4w+a_3w^2+a_2w^3+a_1w^4)(c_1+c_2w+c_3w^2+c_4w^3+c_5w^5).$

Expanding (8*.2), with $m=w^5=D^5+1$, we obtain

(8*.3)
$$\begin{aligned}
a_5c_1 + ma_1c_2 + ma_2c_3 + ma_3c_4 + ma_4c_5 &= 1\\
a_4c_1 + a_5c_2 + ma_1c_3 + ma_2c_4 + ma_3c_5 &= 0\\
a_3c_1 + a_4c_2 + a_5c_3 + ma_1c_4 + ma_2c_5 &= 0\\
a_2c_1 + a_3c_2 + a_4c_3 + a_5c_4 + ma_1c_5 &= 0\\
a_1c_1 + a_2c_2 + a_3c_3 + a_4c_4 + a_5c_5 &= 0.
\end{aligned}$$

The determinant of the system of linear equations (8*.3) equals, interchanging columns with rows,

(8*.4)
$$\Delta = \begin{vmatrix} a_5 & a_4 & a_3 & a_2 & a_1\\ ma_1 & a_5 & a_4 & a_3 & a_2\\ ma_2 & ma_1 & a_5 & a_4 & a_3\\ ma_3 & ma_2 & ma_1 & a_5 & a_4\\ ma_4 & ma_3 & ma_2 & ma_1 & a_5 \end{vmatrix}$$

Now, the reader will verify that the field equation of

$$e^{5n} = a_5 + a_4w + a_3w^2 + a_2w^3 + a_1w^4$$

has exactly the free element=1, since e is a unit, as in case n=3. We thus obtain

(8*.5)
$$(w - D)^{5n} = c_1 + c_2w + c_3w^2 + c_4w^3 + c_5w^4$$

Expanding $(w - D)^{5n}$ we obtain the result. The rational part in the expansion of (8*.5) equals

(8*.6)
$$\sum_{i=0}^{n} (-1)^i D^{5i} w^{5n-5i}.$$

Comparing (8*.6) with c_1 and calculating c_1 from (8*.3), (8*.4), we obtain the identity, with $w^5 = m = D^5 + 1$

(8*.7)
$$\sum_{i=0}^{n} (-1)^i D^{5i} m^{n-i} = \begin{vmatrix} a_5 & a_4 & a_3 & a_2\\ ma_1 & a_5 & a_4 & a_3\\ ma_2 & ma_1 & a_5 & a_4\\ ma_3 & ma_2 & ma_1 & a_5 \end{vmatrix}$$

We substitute the values of a_i, i=1,…,5; from (8*.1) and obtain

(8*.8)
$$\sum_{i=0}^{n} (-1)^i D^{5i} m^{n-i} =$$

$$\begin{vmatrix} \sum_{j=0}^{4}\binom{4}{j}D^jA_0^{(5n+j)} & \sum_{j=0}^{3}\binom{3}{j}D^jA_0^{(5n+1+j)} & \sum_{j=0}^{2}\binom{2}{j}D^jA_0^{(5n+2+j)} & \sum_{j=0}^{1}\binom{1}{j}D^jA_0^{(5n+3+j)} \\ mA_0^{(5n+4)} & \sum_{j=0}^{4}\binom{4}{j}D^jA_0^{(5n+j)} & \sum_{j=0}^{1}\binom{1}{j}D^jA_0^{(5n+3+j)} & \sum_{j=0}^{2}\binom{2}{j}D^jA_0^{(5n+2+j)} \\ m\sum_{j=0}^{1}\binom{1}{j}D^jA_0^{(5n+3+j)} & mA_0^{(5n+4)} & A_0^{(5n+4)} & \sum_{j=0}^{3}\binom{3}{j}D^jA_0^{(5n+1+j)} \\ m\sum_{j=0}^{2}\binom{2}{j}D^jA_0^{(5n+2+j)} & m\sum_{j=0}^{1}\binom{1}{j}D^jA_0^{(5n+3+j)} & m\sum_{j=0}^{4}\binom{4}{j}D^jA_0^{(5n+j)} & \sum_{j=0}^{4}\binom{4}{j}D^jA_0^{(5n+j)} \end{vmatrix}$$

Comparing the powers of w^i (i=1,2,3,4) on both sides of (8*.5) we obtain four more identities with c_i (i=2,3,4,5) calculated from system (8*.3). To have a complete view of (8*.8) the values of $A_0^{(5n+i)}$, (i=0,1,2,3,4) will have to be substituted from (7*.6). This would yield very complicated expressions.

Let us only limit at the difficulties of writing out in full-but not calculating-the determinant (8*.8) in a simple case at D=n=1. We have for the left side of (8*.8)

$$\sum_{i=0}^{1}(-1)^i D^{5i} m^{n-i} = m - D^5 = 1$$

For the right side we calculate

$$A_0^{(5)} = 1, A_0^{(6)} = 5, A_0^{(7)} = 35, A_0^{(8)} = 235, A_0^{(9)} = 1580.$$

Thus the determinant (8*.8) becomes, with the values from (8*.1), viz

$$a_5 = \sum_{j=0}^{4}\binom{4}{j}A_0^{(5+j)} = 1 + 4\cdot 5 + 6\cdot 35 + 4\cdot 235 + 1580 = 2751$$

$$a_4 = \sum_{j=0}^{3}\binom{3}{j}A_0^{(6+j)} = 5 + 3\cdot 35 + 3\cdot 235 + 1580 = 2395$$

$$a_3 = \sum_{j=0}^{2} \binom{2}{j} A_0^{(7+j)} = 35 + 2 \cdot 235 + 1580 = 2085$$

$$a_2 = \sum_{j=0}^{1} \binom{1}{j} A_0^{(8+j)} = 235 + 1580 = 1815$$

$$a_1 = \sum_{j=0}^{0} \binom{0}{j} A_0^{(9+j)} = 1580, \;\; m = 2$$

$$\begin{vmatrix} 2751 & 2395 & 2085 & 1815 \\ 3160 & 2751 & 2395 & 2085 \\ 3630 & 3160 & 2751 & 2395 \\ 4170 & 3630 & 3160 & 2751 \end{vmatrix} = 1$$

Thus formula (8*.8) has been verified for D=n=1. The entries in the right hand determinant become a challenge for n, D>1.

On the combined subject about “Diophantine Equations, Units and Identities” there is not much literature.

Section 4. n-Dimensional Fibonacci numbers and their application from (BGEA)

Bernstein investigated the F(n) function. This function was derived from a special kind of numbers which could well be defined as 3-dimensional Fibonacci numbers. The original Fibonacci numbers should then be called 2-dimensional Fibonacci numbers. This section deals with n-dimensional Fibonacci numbers in a sense to be explained in the sequel. Also, Bernstein derived an interesting identity that was based on 3-dimensional Fibonacci numbers. Carlitz deals with this subject too.

If we remember the original Fibonacci numbers are generated by the formula

$$F(n) = \sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n-i}{i}, n = 1,2,...,$$

then the function

$$F(n) = \sum_{i=0} (-1)^i \binom{n-2i}{i}$$

can be regarded as a generalization of the first, and the author thought that

$$F(n) = \sum_{i=0} (-1)^i \binom{n-ki}{i}, k = 1,2,...,$$

could serve as a k-1-dimensional generalization of the original Fibonaccci numbers, but, regretfully, this consideration led nowhere. From the fact that Fibonacci numbers are derived from the periodic expansion by the Euclidian algorithm of $\sqrt{5}$, there is opened a new horizon for the wanted generalization.
(BGEA) leades to an n-dimensional generalization of Fibonacci numbers.

In this section, the author is introducing the (BGEA) to investigate the various properties and applications of her k-dimensional Fibonacci numbers. It first turns out that these k-dimensional Fibonacci numbers are most useful for a good approximation of algebraic irrationals by rational integers. Further, the author proceeded to investigate higher-degree Diophantine equations and to state identities of a larger magnitude than those investigated before, in an explicit and simple form.

The following formulas were given in Section 1 of this chapter

$$A_s^{(v+n)} = \sum_{k=0}^{n-1} b_k^{(v)} A_s^{(v+k)};\ v = 0,1,...$$

(4.1) $$A_i^{(j)} = \delta_i^j;\ \delta_i^j \text{ the Kroneker delta,}$$

$$i, j = 0,1,..., n-1;\ s = 0,1,...n-1;$$

$$b_k^{(v)} = a_k^{(v)}(D);\ k = 0,1,..., n-1;\ a_0^{(v)} = b_0^{(v)} = 1;$$

$A_s^{(v)}$ are called the matricians of (BGEA) then the three formulas hold:

(4.2) $$\begin{vmatrix} A_0^{(v)} & A_0^{(v+1)} & \cdots & A_0^{(v+n-1)} \\ A_1^{(v)} & A_1^{(v+1)} & \cdots & A_1^{(v+n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n-1}^{(v)} & A_{n-1}^{(v+1)} & \cdots & A_{n-1}^{(v+n-1)} \end{vmatrix} = (-1)^{v(n-1)}$$

(4.3) $$a_s^{(0)} = \frac{\sum_{k=0}^{n-1} a_k^{(v)} A_s^{(v+k)}}{\sum_{k=0}^{n-1} a_k^{(v)} A_0^{(v+k)}},\quad v = 0,1,...;\ s = 0,...,n-1$$

(4.4) $$\prod_{k=1}^{v} a_{n-1}^{(k)} = \sum_{k=0}^{n-1} a_k^{(v)} A_0^{(v+k)}$$

Perron proved the following theorem which, under the conditions of the (BGEA) ($D \geq 1$), becomes

THEOREM 1. The (BGEA) is convergent in the sense that

(4.5) $$\begin{cases} a_s^{(0)} = \dfrac{\lim_{v\to\infty} A_0^{(v)}}{\lim_{v\to\infty} A_0^{(v)}}, s = 1,\dots,n-1 \end{cases}$$

$A_s^{(v)} : A_0^{(v)}$ is called the v th convergent of (BGEA).

In Chapter 2 the author proved

THEOREM 2. If the (BGEA) of $a^{(0)}$ is purely periodic with m=length of the primitive period, then

(4.6) $$\begin{cases} \prod_{k=0}^{m-1} a_{n-1}^{(k)} = \sum_{k=0}^{n-1} a_k^{(m)} A_0^{(m+k)}, \\ \text{is a unit in} Q(w). \end{cases}$$

From (4.6) the formula follows, in virtue of (4.4),

(4.7) $$\left(\prod_{k=0}^{m-1} a_{n-1}^{(k)}\right)^v = \sum_{k=0}^{n-1} a_k^{(m)} A_0^{(vm+k)}, v = 1,2,\dots$$

1*. **Periodic (BGEA)**.

In this section, we construct a periodic (BGEA), with length of primitive period m=1. The fixed vector $a^{(0)}$ must be chosen accordingly, and this may look complicated at first. We prove

THEOREM 3. The (BGEA) of the fixed vector

(1*.1) $$\begin{cases} a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \dots a_s^{(0)}, \dots, a_{n-1}^{(0)}) \\ a_s^{(0)} = \sum_{i=0}^{s} \binom{n-s-1+i}{i} w^{s-i} D^i \\ s = 1,\dots,n-1 \end{cases}$$

is purely periodic and the length of its primitive period m=1.

Proof: We shall first need the formula

(1*.2) $$\sum_{i=0}^{s} \binom{n-s-1+i}{i} = \binom{n}{s}, \quad s = 1,\dots,n-1.$$

This is proved by induction. The proof is to the reader. We have, from (1*.1), the following components of $a^{(0)}$ which we shall use later:

(1*.3) $$a_1^{(0)} = w + (n-1)D;\; a_{n-1}^{(0)} = \sum_{i=0}^{n-1} w^{n-1-i} D^i.$$

Since $w^n - D^n = 1$, we also have

(1*.4) $$\sum_{i=0}^{n-1} w^{n-1-i} D^i = (w-D)^{-1}$$

The vectors $b_i^{(v)}$ (i=1,…,n -1;v=0,1,…) obtained from $a_i^{(v)}(w)$ by the defining rule of (BGEA) are called their corresponding companion vectors. We shall calculate the companion vector $b^{(0)}$ of $a^{(0)}$ and have

$$b_s^{(0)} = \sum_{i=0}^{s} \binom{n-s-1+i}{i} D^{s-i} D^i = D^s \sum_{i=0}^{s} \binom{n-s-1+i}{i},$$

so that, by (1*.2)

(1*.5) $$b_s^{(0)} = \binom{n}{s} D^s,\; s = 1,2,\dots,n-1.$$

Thus,

$$b^{(0)} = \left(\binom{n}{1} D, \binom{n}{2} D^2, \dots, \binom{n}{n-1} D^{n-1} \right).$$

We shall now calculate the vector $a^{(1)}$. From (BGEA) it follows

(1*.6) $$a^{(1)} = (a_1^{(0)} - b_1^{(0)})^{-1} (a_2^{(0)} - b_2^{(0)}, \dots, a_{n-1}^{(0)} - b_{n-1}^{(0)}, 1)$$

From (1*.3), (1*.4), and (1*.5) we obtain:

(1*.7) $$\begin{cases} a_1^{(0)} - b_1^{(0)} = w + (n-1)D - \binom{n}{1} D = w - D, \\ a_{n-1}^{(1)} = (w-D)^{-1} = \sum_{i=0}^{n-1} w^{n-1-i} D^i = a_{n-1}^{(0)} \end{cases}$$

We can prove the relation

(1*.8) $$(a_s^{(0)} - b_s^{(0)})(a_1^{(0)} - b_1^{(0)})^{-1} = a_{s-1}^{(1)},\; s = 2,\dots,n-1$$

Since the proof is elementary, we leave it to the reader.

From (1*.6) it follows that

(1*.9) $$\begin{cases} (a^{(1)} = (a_1^{(0)}, a_2^{(0)}, \dots, a_{n-2}^{(0)}, a_{n-1}^{(0)}) = a^{(0)} \\ a^{(v)} = a^{(0)},\; v = 1,2,\dots \end{cases}$$

This proves Theorem 3.

2*. **Explicit matricians** $A_0^{(v+n)}$

We shall proceed to find an explicit formula for the "zero-degree matricians " $A_0^{(v+n)}$, v=0,1,.., and shall make use, for this purpose, of the defining formula (4.1), and the fact that the (BGEA) is purely periodic with length of the primitive period m=1. Taking into account (1*.5) and (1*.9) we have

(2*.1) $$A_0^{(v+n)} = \sum_{s=0}^{n-1} \binom{n}{s} D^s A_0^{(v+s)}; \quad v = 0,1,...$$

We shall now make use of Euler's generating function. We have

$$\sum_{i=0}^{\infty} A_0^{(i)} x^i = x^0 A_0^{(0)} + \sum_{i=1}^{n-1} A_0^{(i)} x^i + \sum_{i=n}^{\infty} A_0^{(i)} x^i$$

$$= 1 + \sum_{i=0}^{\infty} x^{i+n} \left(A_0^{(i)} + \binom{n}{1} D A_0^{(i+1)} + \binom{n}{2} D^2 A_0^{(i+2)} + ... + \binom{n}{n-1} D^{n-1} A_0^{i+n-1} \right)$$

$$= 1 + x^n \sum_{i=0}^{\infty} A_0^{(i)} x^i + x^{n-1} \sum_{i=0}^{\infty} \binom{n}{1} D A_0^{(i+1)} x^{i+1} + x^{n-2} \sum_{i=0}^{\infty} \binom{n}{2} D^2 A_0^{(i+2)} x^{(i+2)}$$

$$+ ... + x \sum_{i=0}^{\infty} \binom{n}{n-1} D^{n-1} A_0^{(i+n-1)} x^{i+n-1}$$

$$= 1 + \left(x^n + \binom{n}{1} D x^{n-1} + \binom{n}{2} D^2 x^{n-2} + ... + \binom{n}{n-1} D^{n-1} x \right) \sum_{i=0}^{\infty} A_i^{(0)} x^i$$

$$- \left(\binom{n}{1} D x^{n-1} + \binom{n}{2} D^2 x^{n-2} + ... + \binom{n}{n-1} D^{n-1} x \right) = \sum_{i=0}^{\infty} A_0^{(i)} x^i,$$

$$\left[1 - \left(x^n + \sum_{k=1}^{n-1} \binom{n}{k} D^k x^{n-k} \right) \right] \sum_{i=0}^{\infty} A_0^{(i)} x^i = 1 - \sum_{k=1}^{n-1} \binom{n}{k} D^k x^{n-k},$$

$$\sum_{i=0}^{\infty} A_0^{(i)} x^i = \frac{1 - \left(x^n + \sum_{k=1}^{n} \binom{n}{k} D^k x^{n-k} \right) + x^n}{1 - \left(x^n + \sum_{k=1}^{n-1} \binom{n}{k} D^k x^{n-k} \right)} = \frac{x^n}{1 - \sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k}} + 1$$

$$A_0^{(0)} + \sum_{i=0}^{\infty} A_0^{(i)} x^i = x^n \sum_{t=0}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} D^k x^{n-k} \right)^t + 1$$

$$xA_0^{(1)} + A_0^{(2)}x^2 + \dots + A_0^{(n-1)}x^{n-1} + \sum_{i=n}^{\infty} A_0^i x^i = x^n \sum_{t=0}^{\infty}\left(\sum_{k=0}^{n-1}\binom{n}{k}D^k x^{n-k}\right)^t,$$

For x sufficiently small. Thus, since $A_0^1 = \dots = A_0^{(n-1)} = 0$, we have

$$\sum_{i=n}^{\infty} A_0^i x^i = x^n \sum_{t=0}^{\infty}\left(\sum_{k=0}^{n-1}\binom{n}{k}D^k x^{n-k}\right)^t$$

$$\sum_{i=0}^{\infty} A_0^{(n+i)} x^{n+i} = x^n \sum_{t=0}^{\infty}\left(\sum_{k=0}^{n-1}\binom{n}{k}D^k x^{n-k}\right)^t$$

(2*.2) $$\sum_{i=0}^{\infty} A_0^{(n+i)} x^i = \sum_{t=0}^{\infty}\left(\sum_{k=0}^{n-1}\binom{n}{k}D^k x^{n-k}\right)^t$$

and comparing coefficients of powers x^v on both sides of (2*.2), we obtain

(2*.3) $$A_0^{(v+n)} = \sum_{ny_1+(n-1)y_2+\dots+2y_{n-1}+y_n=v}\binom{y_1+y_2+\dots y_n}{y_1, y_2, \dots, y_n}\prod_{k=0}^{n-1}\left(\binom{n}{k}D^k\right)^{y_{k+1}}$$

or

(2*.4) $$A_0^{(v+n)} = \sum_{\substack{\sum_{i=0}^{n-1}(n-i)y_{i+1}=v \\ v=0,1,\dots}}\binom{y_1+y_2+\dots+y_n}{y_1, y_2, \dots, y_n} D^{\sum_{j=1}^{n-1} j y_{j+1}} \prod_{k=0}^{n-1}\binom{n}{k}^{y_{k+1}}$$

Formula (2*.4) looks very complicated. $A_0^{(v+n)}$ can also be calculated by the recurrency relation (4.1). It is conjectured thet it is easier to do so by formula (2*.4), and would be a challenging computer problem.

3*. **Matricians of degree s, s=1,2,…,n-1**

Here, we shall express “s-degree matricians”

$$A_s^{(v)},\ s=1,\dots,n-1$$

by means of zero=degree matricians. This is not an easy task. Now we shall prove very important theorem.

THEOREM 4. The s-degree matricians are expressed through the zero-degree matricians by means of the relations

(3*.1) $$A_s^{(v+n-1)} = \sum_{k=0}^{s}\binom{n}{k}D^k A_0^{(v+n-s+k-1)}, v=0,1,\dots;\ \ s=1,\dots,n-1.$$

Proof: From formula (4.6) it follows that

(3*.2) $$a_0^{(0)} \sum_{k=0}^{n-1} a_k^{(0)} A^{(v+k)} = \sum_{k=0}^{n-1} a_k^{(0)} A_0^{(v+k)}, s=1,2,...,n-1;\ v=0,1,...$$

Or, writing a_i for $a_i^{(0)}$, i=0,..,n -1, and substituting their values from (1*.1), we obtain

(3*.3) $$\sum_{i=0}^{s} \binom{n-s-1+i}{i} w^{s-i} D^i \sum_{k=0}^{n-1} a_k A_0^{(v+k)} = \sum_{k=0}^{n-1} a_k A_s^{(v+k)}$$

We shall now compare coefficients of w^{n-1} on both sides of (3*.3). The power of w^{n-1} appears, on the right side only in

$$a_{n-1} = w^{n-1} + Dw^{n-2} + ... + D^{n-1}$$

and its coefficients are

(3*.4) $$A_s^{(v+n-1)}$$

So the whole problem is to find the coefficient of w^{n-1} on the left side, and this problem is *the* problem. We shall with the first power of w in a_s, which is w^s (in the left side). Now in

$$\sum_{k=0}^{n-1} a_k A_0^{(n+k)}$$

we have to look for those $a_k^{'}s$ which have the powers w^{n-s-1}; this appears in

$$a_{n-s-1} \text{(first term, coefficient=} A_0^{(v+n-s-1)})$$

$$a_{n-s} \text{(second term, coefficient=} \binom{s}{1} DA_0^{(v+n-s)})$$

$$a_{n-s+1} \text{(third term, coefficient =} \binom{s}{2} D^2 A_0^{(v+n-s+1)})$$

M

$$a_{n-1} \text{((1+s)th term, coefficient=} \binom{s}{s} D^s A_0^{(v+n-1)})$$

Thus, we have obtained the partial sum of coefficients of w^{n-1} in the left side.

$$A_0^{(v+n-s-1)} + \binom{s}{1} DA_0^{(v+n-s)} + \binom{s}{2} D^2 A_0^{(v+n-s+1)} + ... + \binom{s}{s} D^s A_0^{(v+n-1)}$$

Now the next power of a_s on the left side is w^{s-1} with coefficient

$$\binom{n-s-1+1}{1} D = \binom{n-s}{1} D$$

To obtain w^{n-1}, w^{s-1} must be multiplied by n-s, so we must start with the first term of a_{n-s}, the second term of a_{a-s+1},.., etc. Compared with the previous sum, to be replaced by s-1. The sum will then be multiplied by $\binom{n-s}{1}D$, and the member of summands will be smaller by one. We then obtain the partial sum:

$$\binom{n-s}{1}D\left[A_0^{(v+n)}+\binom{s-1}{1}DA_0^{(v+n-s+1)}+\binom{s-1}{2}D^2A_0^{(v+n-s+2)}+\ldots+\binom{s-1}{s-1}D^{s-1}A_0^{(v+n-1)}\right]$$

Proceeding in this way, we obtained the partial sums: