$$\left\{\begin{array}{l} A_0^{(v+n-s-1)} + \binom{s}{1} DA_0^{(v+n-s)} + \binom{s}{2} D^2 A_0^{(v+n-s+1)} + \binom{s}{3} D^3 A_0^{(v+n-s+2)} + \binom{s}{4} D^4 A_0^{(v+n-s+3)} + \ldots \\ \qquad + \binom{n-s}{1} DA_0^{(v+n-s)} + \binom{n-s}{1}\binom{s-1}{1} D^2 A_0^{(v+n-s+1)} + \binom{n-1}{1}\binom{s-1}{2} D^3 A_0^{(v+n-s+2)} + \binom{n-s}{1}\binom{s-1}{3} D^4 A_0^{(v+n-s+3)} + \\ \qquad\qquad + \binom{n-s+1}{2}\binom{s-2}{0} D^2 A_0^{(v+n-s+1)} + \binom{n-s-1}{2}\binom{s-2}{1} D^3 A_0^{(v+s-n+2)} \end{array}\right.$$

$+\ldots+$

Thus the general term in the sum of coefficients of w^{n-1} on the left side of (3*.3) which contains $D^k A_0^{(v+n-s-1+k)}$ as a constant factor has the form, adding up in (4*.5) the column with this factor,

(3*.6) $$\sum_{i=0}^{k}\binom{n-s-1+i}{i}\binom{s-i}{k-i}D^k A_0^{(v+n-s-k+k)}$$

The following formula is well known:

(3*.7) $$\sum_{i=0}^{k}\binom{n-s-1+i}{i}\binom{s-i}{k-i}=\binom{n}{k}$$

which becomes formula (1*.2) for k=2. Now, since in

$$a_s=\sum_{i=0}^{s}\binom{n-s-1+i}{i}w^{s-i}D^i$$

The exponent of D sums from i=0 to i=s, we have, finally,

$$A_s^{(v+n-1)}=\sum_{k=0}^{s}\binom{n}{k}D^k A_0^{(v+n-1-s+k)}$$

which is formula (3*.1) and proves Theorem 4. From formula (3*.1), we have the single cases

(3*.8) $$A_1^{(v+n-1)}=A_0^{(v+n-2)}+\binom{n}{1}DA_0^{(v+n-1)}$$

and

(3*.9) $$A_{n-1}^{(v+n-1)}=A_0^{(v+n)}$$

(3*.9) is a very surprising relation and will be applied in the next section. Similarly,

(3*.10) $$A_2^{(v+n-1)}=A_0^{(v+n-3)}+\binom{n}{1}DA_0^{(v+n-2)}+\binom{n}{2}D^2A_0^{(v+n-s)},\text{etc.}$$

4*. Approximation of irrationals by rationals.

We shall investigate especially the case D=1, but produce first formulas for any value of D. We obtain from (3*.8) and (4.3)

$$a_1^{(0)}=\frac{\lim_{v\to\infty}A_1^{(v+n-1)}}{\lim_{v\to\infty}A_0^{(v+n-1)}}=\lim_{v\to\infty}=\frac{A_0^{(v+n-2)}+nDA_0^{(v+n-1)}}{A_0^{(v+n-1)}}$$

$$w+(n-1)D=nD+\lim_{v\to\infty}\frac{A_0^{(v+n-2)}}{A_0^{(v+n-1)}}$$

(4*.1) $$w = D + \lim_{v\to\infty} \frac{A_0^{(v+n-2)}}{A_0^{(v+n-1)}} = D + \lim_{v\to\infty} \frac{A_0^{(v+n-1)}}{A_0^{(v+n)}}$$

For D=1, w=$\sqrt[n]{2}$, and from (2*.4) and (4*.1) we obtain the approximation formula

$$\sqrt[n]{2} \approx \left\{ \frac{\sum_{\sum (n-i)y_{i+1}=v, i=0,\dots,n-1} \binom{y_1+y_2+\dots+y_n}{y_1, y_2,\dots,y_n} \prod_{k=0}^{n-1} b^{y_{k+1}}}{\sum_{\sum (n-i)y_{i+1}=v+1, i=0,\dots,n-1} \binom{y_1+y_2+\dots+y_n}{y_1, y_2,\dots,y_n} \prod_{k=0}^{n-1} b^{y_{k+1}}} \right.$$

$$b_k = \binom{n}{k}, k = 0,\dots, n-1; b_0 = 1$$

The approximation are not very close, and we would have to continue a few steps further to get a closer approximation. Formula (3*.9), surprisingly simple as it is, does not yield any news. It enables us to calculate w^{n-1} by means of the powers, w_k, k=1,…,n-2.

We have approximately, expanding $\sqrt[n]{2} = (1+1)^{1/n}$ by the binomial series,

$$\sqrt[n]{2} \approx 1 + \frac{1}{n}$$

According to our approximation formula (4*.1) with D=1,

$$\sqrt[n]{2} = w \approx 1 + \frac{A_0^{(n)}}{A_0^{(n+1)}};$$

$$A_0^{(n+1)} = A_0^{(1)} + \binom{n}{1} A_0^{(2)} + \dots + \binom{n}{n-1} A_0^{(n)} = \binom{n}{n-1} A_0^{(n)} = nA_0^{(n)} = n$$

since $A_0^{(n)} = A_0^{(0)} + A_0^{(1)} + \dots + \binom{n}{n-1} A_0^{(n-1)} = A_0^{(0)} = 1, \sqrt[n]{2} \approx 1 + \frac{1}{n}$, as should be.

5*. **Diophantine equations**.

We shall construct two types of Diophantine equations of degree n in n unknowns and state their explicit solutions, which are infinite in number. We have from (4.2)

(5*.1) $$\begin{vmatrix} A_0^{(v+n)} & A_0^{(v+n+1)} & A_0^{(v+n+2)} & \dots A_0^{(v+n+n-1)} \\ A_1^{(v+n)} & A_1^{(v+n+1)} & A_1^{(v+n+2)} & \dots A_1^{(v+n+n-1)} \\ \dots & \dots & \dots & \dots \\ A_{n-1}^{(v+n)} & A_{n-1}^{(v+n-1)} & A_{n-1}^{(v+n+2)} & A_{n-1}^{(v+n+n-1)} \end{vmatrix} = (-1)^{(n-1)v}, \text{ v=0,1}$$

Substituting in (5*.1) the values of $A_s^{(t)}$ from (3*.1) we obtain, after simple row rearrangements,

$$(5^*.2)\qquad \begin{vmatrix} A_0^{(v+n)} & A_0^{(v+n+1)} & A_0^{(v+n+2)} & \dots & A_0^{(v+n+n-1)} \\ A_0^{(v+n-1)} & A_0^{(v+n)} & A_0^{(v+n+1)} & \dots & A_0^{(v+n+n-2)} \\ A_0^{(v+n-2)} & A_0^{(v+n-1)} & A_0^{(v+n)} & \dots & A_0^{(v+n+n-3)} \\ \dots & \dots & \dots & \dots & \dots \\ A_0^{(v+3)} & A_0^{(v+4)} & A_0^{(v+5)} & \dots & A_0^{(v+n+2)} \\ A_0^{(v+2)} & A_0^{(v+3)} & A_0^{(v+4)} & \dots & A_0^{(v+n+1)} \\ A_0^{(v+1)} & A_0^{(v+2)} & A_0^{(v+3)} & \dots & A_0^{(v+n)} \end{vmatrix} = (-1)^{(n-1)v}$$

We introduce the notation

$$(5^*.3)\qquad X_{v,k} = A_0^{(v+k)},\ k = 1,2,\dots,n$$

$$(5^*.4)\qquad \begin{cases} A_0^{(v+k)} = A_0^{(v+k-n)} + b_1 A_0^{(v+k-n+1)} + b_2 A_0^{(v+k+2-n)} + \dots + b_{n-1} A_0^{(v+k-1)} \\ b_k = \binom{n}{k} D^k,\ k = 0,1,\dots,n-1,\ v = 1,2,\dots \end{cases}$$

We introduce these notations in (5*.2) and then make the following manipulations in this determinant.

From the first row we subtract the b_1 multiple of the first row from below, then the b_2 multiple of second row from below,…, then the b_kth multiple of the *k*th row from below, k=1,…,n-1.

Then (5*2) takes the form, in virtue of (5*.3) and (5*.4)

$$(5^*5)\qquad \begin{vmatrix} X_{v,n} - \sum_{k=1}^{n-1} b_k X_{v,k} & X_{v,1} & X_{v,2} & \dots & X_{v,n-1} \\ A_0^{(v+n-1)} & A_0^{(v+n)} & A_0^{(v+n-1)} & \dots & A_0^{(v+n+n-2)} \\ A_0^{(v+n-2)} & A_0^{(v+n-1)} & A_0^{(v+n)} & \dots & A_0^{(v+n+n-3)} \\ \dots & \dots & \dots & \dots & \dots \\ A_0^{(v+2)} & A_0^{(v+3)} & A_0^{(v+4)} & \dots & A_0^{(v+n-1)} \\ X_{v,1} & X_{v,2} & X_{v,3} & \dots & X_{v,n} \end{vmatrix} = (-1)^{(n-1)v}$$

We subtract from the second row the b_2 multiple of the first row from below, the b_3 multiple of the second row from below,…, the b_k multiple of the (k-1)th row from below; the determinant (5*.5) then takes the form (k=2,…,n-2):

(5*.6)

$$\left\{\left|\begin{matrix} X_{v,n}-\sum_{k=1}^{n-1} b_k X_{v,k} & X_{v,1} & X_{v,2} & X_{v,3} & \cdots & X_{v,n-1} \\ X_{v,n-1}-\sum_{k=1}^{n-2} b_{k+1} X_{v,k} & X_{v,n}-\sum_{k=1}^{n-2} b_{k+1} X_{v,k+1} & X_{v,1}+b_1 X_{v,2} & X_{v,2}+b_1 X_{v,3} & \cdots & X_{v,n-2}+b_1 X_{v,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ V_{v,1} & X_{v,2} & X_{v,3} & X_{v,4} & \cdots & X_{v,n} \end{matrix}\right|\right.$$

$$= (-1)^{(n-1)v}$$

Continuing this process by another step, the third row of determinant (5*.6) will have the form

$$X_{v,n-2}-\sum_{k=1}^{n-3} b_{k+2} X_{v,k} X_{v,n-1}-\sum_{k=1}^{n-3} b_{k+2} X_{v,k+1} X_{v,n}-\sum_{k=1}^{n-3} b_{k+2} X_{v,k+2}$$
$$X_{v,1}+b_1 X_{v,2}+b_2 X_{v,3} X_{v,2}+b_1 X_{v,3}+b_2 X_{v,4} \ldots$$
$$X_{v,n-3}+b_1 X_{v,n-2}+b_2 X_{v,n-1}$$

Generally we subtract from the th row in (5*.2) the b_i multiple of the first row from below, then the b_{i+1} multiple of the second row from below,..,the b_{n-1} multiple of the (n-i)th row from below (i=1,..,n -1). The reader can verify, that by these operations the determinant (5*.2) transforms into one containing only the unknowns $X_{v,i}$ (i=1,..,n), which yields the Diophantine equation of degree n in these unknowns.

6*. **More diophantine equations**.

The (BGEA) of $a^{(0)}$ is purely periodic with length of the primitive period m=1. Since

$$a_{n-1}^{(0)}=\sum_{i=0}^{n-1}\binom{n-1-(n-1)+i}{i} w^{n-1-i} D^i=\sum_{i=0}^{n-1} w^{n-1-i} D^i$$

we have by Theorem 2 and formula (4.7)

(6*.1) $$(w^{n-1}+Dw^{n-2}+\ldots+D^{n-1})^v=\sum_{i=0}^{n-1} a_i^{(0)} A_0^{(v+i)}, \; v=1,2,\ldots$$

We find the norm of $(w^{n-1}+Dw^{n-2}+\ldots+D^{n-1})^v$. We have

(6*.2) $$\begin{cases} D^n-w^n=-1, \\ D^n-w^n=-\prod_{k=0}^{n-1}(D-\rho_k w)=-N(D-w), \\ \rho_k=e^{2\pi ik/n}, k=0,1,\ldots,n-1 \end{cases}$$

But $w^{n-1}+Dw^{n-2}+\ldots+D^{n-1}=-(D-w)^{-1}$; hence, (6*.3) $N[(w^{n-1}+Dw^{n-2}+\ldots+D^{n-1})^{v}]=(-1)^{(n-1)v}, v=1,2,..; \; n=2,3,\ldots.$

We have

$$\sum_{i=0}^{n-1} a_i^{(0)} A_0^{(v+i)} = A_0^{(v)} + \left[w+\binom{n-1}{1}D\right]A_0^{(v+1)} + \left[w^2+\binom{n+2}{2}Dw+\binom{n-1}{2}D^2\right]A_0^{(v+2)}$$
$$+\left[w^3+\left[\begin{matrix}n-3\\1\end{matrix}\right]w^2D+\binom{n-2}{2}wD^2+\binom{n-1}{3}D^3\right]A_0^{(v+3)}+\ldots$$
$$+\left[w^{n-1}+\binom{1}{1}w^{n-1}D+\binom{2}{2}w^{n-2}D^2+\ldots+\binom{n-1}{n-1}D^{n-1}\right]A_0^{(v+n-1)}$$

Denoting

(6*.4) $$X_{v,k}=\sum_{s=0}^{n-1-k}\binom{n-1-k}{s}A_0^{(v+s+k)}D^s, \; k=0,1,\ldots,n-s$$

This $X_{v,k}$ is not the $X_{v,k}$ from (5*.3). We have from (6*.1)

(6*.5) $$(w^{n-1}+Dw^{n-2}+\ldots+D^{n-1})^v=\sum_{k=0}^{n-1}X_{v,k}w^k=e^v, \text{e a unit}$$

We shall find the field equation of $\sum_{k=0}^{n-1}X_{v,k}w^k$

The free member of it is the norm of e^v, and since e^v, is a unit with norm $(-1)^{(n-1)v}$, according to (6*.3), we find easily, by known methods, that

(6*.6) $$\begin{vmatrix} X_{v,0} & X_{v,1} & X_{v,2} & \ldots & X_{v,n-2} & X_{v,n-1} \\ mX_{v,n-1} & X_{v,0} & X_{v,1} & \ldots & X_{v,n-3} & X_{v,n-2} \\ mX_{v,n-2} & mX_{v,n-1} & X_{v,0} & \ldots & X_{v,n-4} & X_{v,n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ mX_{v,2} & mX_{v,3} & mX_{v,4} & \ldots & X_{v,0} & X_{v,1} \\ mX_{v,1} & mX_{v,2} & mX_{v,3} & \ldots & mX_{v,n-1} & X_{v,0} \end{vmatrix}=(-1)^{(n-1)v}$$

It is not difficult to see that, in the case n=2m+1 (m=1,2,…), the highest powers of the *n* unknowns of the discriminant (6*.6) as

$$X_{v,0}^n, mX_{v,1}^n, m^2X_{v,2}^n, \ldots, m^{n-1}X_{v,n-1}^n,$$

while the last unknown, $X_{v,n-1}$ does not have the exponent *n*, but a smaller one. In the case *n=2m* (m=1,2,…) these n -1 powers are the same, but with alternating signs, viz.,

$$X_{v,0}^n, -mX_{v,1}^n, +m^2X_{v,2}^n, \ldots.$$

In the case *n*=2, the expanded discriminate (6*.6) had the form

$$X_v^2-mY_v^2=\pm1,$$

and the case *n*=3, it had the form

$$X^3 + mY^3 + m^2Z^3 - 3mXYZ = 1$$

The first is Euler-Pell's Equation.

7*. **Identities and units**.

We return to formulas (6*.4) and (6*.5), and have

(7*.1)
$$\begin{cases} X_{nv,k} = \sum_{s=0}^{n-1-k} \binom{n-1-k}{s} A_0^{(vn+s+k)} D^s, \quad k = 0,1,\dots,n-1 \\ (w^{n-1} + Dw^{n-2} + \dots + D^{n-1}) = \sum_{k=0}^{n-1} X_{nv,k} w^k \end{cases}$$

We compare powers of w^k (k=0,1,..,n -1) on both sides of (7*.1) and take into consideration that $w^{nt} = m^t = (D^n + 1)^t$. We have, looking for the rational part of the right side, k=0, and the value of the right side equals $X_{nv,0}$, and by (6*.4)

(7*.2)
$$X_{nv,0} = \sum_{s=0}^{n-1} \binom{n-1}{s} A^{(nv+s)} D^s, \quad v = 0,1\dots$$

On the left side, we have to look for the coefficients of w^n. Since the highest power in the expression $(w^{n-1} + Dw^{n-2} + \dots + D^{n-1})^{nv}$ is n(n-1)v, we have the expression

(7*.3)
$$\sum_{\substack{\sum_{i=1}^{n-1}(n-i)y_i = sn \le n(n-1)v, \\ \sum_{i=1}^{n-1} iy_{i+1} = n(n-1)v - sn, s=0,1,\dots,(n-1)v}} \binom{y_1 + y_2 + \dots + y_n}{y_1, y_2, \dots, y_n} w^{\sum_{i=1}^{n-1}(n-i)y_i} D^{\sum_{i=1}^{n-1} iy_{i+1}} = X_{nv}$$

We want to obtain in this way the rational part of $(w^{n-1} + Dw^{n-2} + \dots + D^{n-1})^{nv}$.

At the same time $\sum_{i=1}^{n-1} iy_{i+1}$ is the sum of the exponents of the powers of y_{i+1} (i=1,..,n - 1). Since in every summand of $w^{n-1} + Dw^{n-2} + \dots + D^{n-1}$ the sum of the exponents of $D^i w^{n-1-i}$ (i=0,1,..,n -1) is n-1, and the highest exponent in the expansion if n(n-1)v, we have that $\sum_{i=1}^{n-1}(n-i)y_i + \sum_{i=1}^{n-1} iy_{i+1} = n(n-1)v$, which explains the left side of (7*.3). We further have

(7*.4)
$$\sum_{i=1}^{n-1} [(n-i)y_i + iy_{i+1}] = n(n-1)v,$$

so that $y_1 + y_2 + ... + y_n = nv$

Now, taking into account that the exponent of w under the summation sign in (7*.3) equals sn, $w^{sn} = m^s$, and D^n=m-1, formula (7*.3) takes the form

(7*.5) $$\begin{cases} \sum\limits_{\sum_{i=1}^{n-1}(n-i)y_i = sn} \binom{nv}{y_1, y_2, ..., y_n} m^s (m-1)^{(n-1)v-s} = X_{nv,0} \\ = \sum\limits_{k=0}^{n-1} \binom{n-1}{k} D^k A_0^{(nv+k)}; \quad s = 0,1,...,(n-1)v; v = 0,1,...; y_1 + y_2 + ... + y_n = nv \end{cases}$$

(7*.5) is an interesting combinatorial identity.

From (7*.1), n-1 more identities can be obtained by comparing the coefficients of the powers w^i, i=1,..,n -1, on both sides of (7*.1). The identities have a somewhat complicated from; however, they will express the coefficients of w^t, t=1,..,n -1, in the expansion of $(w^{n-1} + Dw^{n-2} + ... + D^{n-1})^{nv}$ with $w^n = m = D^n + 1$.

$$\begin{cases} \sum\limits_{\substack{\sum_{i=1}^{n-1}(n-i)y_i = sn+t \le n(n-1)v \\ \sum_{i=1}^{n-1} iy_{i+1} = n(n-1)v-(sn+t)}} \binom{nv}{y_1, y_2, ..., y_n} m^s (m-1)^{(n-1)v-s-1} D^{n-t} = X_{nv,t} \\ = \sum\limits_{j=0}^{n-1-t} \binom{n-1-t}{j} A_0^{(nv+j+t)} D^j; \qquad j = 0,1,...,(n-1)v-1; t = 1,...,n-1 \end{cases}$$

We wish to explain the appearance of the factor D^{n-t} under the summation sign on the left side of (7*.6). The power of D in the expansion of $(w^{n-1} + Dw^{n-2} + ... + D^{n-1})^{nv}$ equals

$$\begin{aligned} \sum_{i=1}^{n-1} iy_{i+1} &= n(n-1)v - (sn+t) \\ &= n(n-1)v - sn - n + (n-t) \\ &= n[(n-1)v - s - 1] + n - t. \end{aligned}$$

Thus, the power of D equals

$$(D^n)^{n(n-1)v-s-1} \cdot D^{n-t}, \text{ with } D^n = m-1.$$

The power of w is

$$\sum_{i=1}^{n-1} (n-i)y_i = sn + t = (w^n)^s w^t = m^s w^t$$

so m^s is the coefficient of w^t as desired.

Section 5. (BGEA) and sums of some infinite series

In this section the author will use some of previous results from (BGEA), in the theory of units in algebraic number fields.

Starting with a unit $e=\frac{(w-D)^n}{d}$ from (BGEA) and powers of this unit in $Q(w)$, $w=\sqrt[n]{D^n+d}$, $d\in N$, $d\in Z$, $d|D$ and $w^n=m$, we will evaluate the sums of some infinite series.

THEOREM 1. If $e=\frac{(w-D)^n}{d}$ in $Q(w)$, $w=\sqrt[n]{D^n+d}$, $D\in N$, $d\in Z$, $d|D$ and $w^n=m$.

(5.1)
$$A=\begin{bmatrix} s_{0,1} & ms_{n-1,1} & \dots & ms_{2,1} & ms_{1,1} \\ s_{1,1} & s_{0,1} & \dots & ms_{3,1} & ms_{2,1} \\ \dots & \dots & \dots & \dots & \dots \\ s_{n-2,1} & s_{n-3,1} & \dots & s_{0,1} & ms_{n-1,1} \\ s_{n-1,1} & s_{n-2,1} & \dots & s_{1,1} & s_{0,1} \end{bmatrix}$$

where

(5.2)
$$\begin{cases} s_{0,1}=\dfrac{m+1+(-1)^n D^n}{d} \\ s_{1,1}=(-1)^{n-i}\dbinom{n}{n-i}\dfrac{D^{n-i}}{d} \end{cases}$$

$n\in N$, $d\in Z$, $D\in N$, $d\,|\,D$, $n>2$, $m=D^n+d$
then

(5.3)
$$\begin{cases} \displaystyle\sum_{j=0}^{\infty}\binom{(j+1)n-1}{jn}\left(\frac{D^n}{m}\right)^j=\frac{m}{d}|A_1| \\ \displaystyle\sum_{j=1}^{\infty}\binom{(j+1)n-i-1}{jn}\left(\frac{D^n}{m}\right)^j=\frac{m}{d}|A_{i+1}|,\quad (i=1,2,...,n) \end{cases}$$

where $|A_1|$ is $|A|$ with the first column replaced by the vector (1,0,..,0) and $|A_{i+1}|$ is $|A|$ with the i-th column replaced by the vector (1,0,..0).

Proof : Suppose

(5.4)
$$\begin{cases} e = s_{0,1} + s_{1,1}w + ... + s_{n-1,1}w^{n-1}; & s_i \in Z \\ e^{-1} = t_{0,1} + t_{1,1}w + ... + t_{n-1,1}w^{n-1}; & t_i \in Z \\ \quad i = 0,1,...,n-1 \end{cases}$$

We perform ee^{-1} reducing the powers of w, knowing that $w^n = D^n + d = m$ and obtain the system of n equations:

(5.5)
$$\begin{cases} 1 = s_{0,1}t_{0,1} + m(s_{n-1,1}\cdot t_{1,1} + s_{n-2,1}\cdot t_{2,1} + ... + s_{1,1}\cdot t_{n-1,1}) \\ 0 = (s_{1,1}t_{0,1} + s_{0,1}\cdot t_{1,1}) + m(s_{n-1,1}\cdot t_{0,1} + s_{n-2,1}\cdot t_{2,1} + ... + s_{2,1}\cdot t_{n-1,1}) \\ 0 = (s_{2,1}t_{0,1} + s_{1,1}\cdot t_{1,1} + + s_{0,1}t_{2,1}) + m(s_{n-1,1}\cdot t_{3,1} + s_{n-2,1}\cdot t_{4,1} + ... + s_{3,1}\cdot t_{n-1,1}) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 0 = (s_{n-2,1}\cdot t_{0,1} + s_{n-3,1}\cdot t_{1,1} + ... + s_{0,1}\cdot t_{n-2,1}) + m(s_{n-1,1}\cdot t_{n-1,1}) \\ 0 = (s_{n-1,1}\cdot t_{0,1} + s_{n-2,1}\cdot t_{1,1} + ... + s_{0,1}\cdot t_{n-1,1}) \end{cases}$$

Taking the $t_{i,1}$'s as unknowns, the determinant of the system is $|A|$, where A is as in (5.1).

Since $w^n = m$, Q(w) has a basis $1, w, .., w^{n-1}$, so that any algebraic number in Q(w) has the form $\sigma = x_1 + x_2 w + ... + x_n w^{n-1}$, $x_i \in Q$ (i=1,..,n-1). The norm of σ is a polynomial in the x_i and N(σ)=1 so we have $(-1)^{n-1}S_n = m$, $S_j = 0$ for j=n, σ=e and S_n and S_j are the elementary symmetric functions in $D_0, D_1, .., D_n$ and

$(-1)^n S_n = (-1)^n D_0 D_1 ... D_{n-1} = d$

In this case

(5.6)
$$N(e) = \begin{vmatrix} s_{0,1} & s_{1,1} & ... & s_{n-2,1} & s_{n-1,1} \\ ms_{n-1,1} & s_{0,1} & ... & s_{n-3,1} & s_{n-2,1} \\ ... & ... & ... & ... & ... \\ ms_{2,1} & ms_{3,1} & ... & s_{0,1} & s_{1,1} \\ ms_{1,1} & ms_{2,1} & ... & ms_{n-1,1} & s_{0,1} \end{vmatrix}$$

which is the transpose of the |A| with A in (5.1).

Also, $N(e) = N\left[\frac{(w-D)^n}{d}\right] = \left[N\left(\frac{w-D}{d}\right)\right]^n$ and

$$N(w-D) = \begin{vmatrix} -D & 1 & 0 & 0 & 0 \\ 0 & -D & 1 & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & -D & 1 \\ m & 0 & 0 & 0 & -D \end{vmatrix} = (-D)(-1)^{n-1}D^{n-1} + (-1)^{n-1}m = (-1)^{n-1}d$$

and N(e)=1.

Now we apply Cramer's rule to solve for $t_{0,1}$ in (5.1) and

(5.7) $$t_{0,1} = \begin{vmatrix} 1 & ms_{n-1,1} & \ldots ms_{2,1} & ms_{1,1} \\ 0 & s_{0,1} & \ldots ms_{3,1} & ms_{2,1} \\ \ldots & \ldots & \ldots \; \ldots & \ldots \\ 0 & s_{n-3,1} & \ldots \; s_{0,1} & ms_{n-1,1} \\ 0 & s_{n-2,1} & \ldots \; s_{1,1} & s_{0,1} \end{vmatrix} \text{ and } e = \frac{(w-D)^n}{d}$$

if expanded and compared with (5.4) gives (5.2).

$$\begin{cases} s_{0,1} = \dfrac{m + (-1)^n D^n}{d} \\ s_{i,1} = (-1)^{n-i} \dbinom{n}{n-i} \dfrac{D^{n-i}}{d} \end{cases} \quad (i=1,..,n\text{ -}1)$$

(5.7) solves for $t_{0,1}$ in terms of the parametric forms for the (5.2)
Another way to write $t_{0,1}$ is:

$$e^{-1} = d(w-D)^{-n} = \frac{d}{m}\left(1 - \frac{D}{w}\right)^{-n} \text{ and since } \left|\frac{D}{w}\right| < 1, \text{we have}$$

$$e^{-1} = \frac{d}{m}\sum_{j=0}^{\infty}\binom{n+j-1}{j}\left(\frac{D}{w}\right)^j$$

if $(k-1)^n < j < kn$, then $\dfrac{1}{w^j} = \dfrac{w^{kn-j}}{m^k}$ and $0 \le kn - j \le n-1$.

Let regroup the terms of our infinite series in the powers of w.

(5.8) $$\begin{aligned} e^{-1} &= \frac{d}{m}\sum_{j=0}^{\infty}\binom{(j+1)n-1}{jn}\frac{D^{jn}}{m^j} + \left(\frac{d}{m}\sum_{j=0}^{\infty}\binom{jn+2n-2}{jn+n-1}\frac{D^{jn+n-1}}{m^{j+1}}\right)w + \\ &+ \ldots + \left(\frac{d}{m}\sum_{j=0}^{\infty}\binom{jn+n+1}{jn+2}\frac{D^{jn+2}}{m^{j+1}}\right)w^{n-2} + \left(\frac{d}{m}\sum_{j=0}^{\infty}\binom{jn+n}{jn+1}\frac{D^{jn+1}}{m^{j+1}}\right)w^{n-1} \end{aligned}$$

From (5.8) we have

(5.9) $$t_{0,1} = \frac{d}{m}\sum_{j=0}^{\infty}\binom{(j+1)n-1}{jn}\left(\frac{D^n}{m}\right)^j$$

(5.9) gives a parametric family of infinite series and an explicit form of their sums as

$$\sum_{j=0}^{\infty}\binom{(j+1)n-1}{jn}\left(\frac{D^n}{m}\right)^j = \frac{m}{d}|A_1|$$

or $\sum_{j=0}^{\infty}\binom{(j+1)n-1}{jn}\left(\frac{m}{D^n}\right)^j = \frac{m}{d}\left|A_1\right|$ as (5.3) where $|A_1|$ is $|A|$ with the first column replaced by the vector (1,0,..,0), and $s_{0,1}$ and $s_{i,1}$ as in (5.2).

EXAMPLE **for n=4.** $m = w^4 = D^4 + d$

$$\sum_{j=0}^{\infty}\left(\binom{4j+3}{4j}\left(\frac{D^4}{m}\right)^j\right) = \frac{m}{d}\left|A_1\right|$$

$$\left|A_1\right| = \begin{vmatrix} 1 & ms_{3,1} & ms_{2,1} & ms_{1,1} \\ 0 & s_{0,1} & ms_{3,1} & ms_{2,1} \\ 0 & s_{1,1} & ms_{0,1} & ms_{3,1} \\ 0 & s_{2,1} & s_{1,1} & s_{0,1} \end{vmatrix}$$

$$s_{0,1} = \frac{m + (-1)^4 D^4}{d} \text{ and}$$

$$s_{i,1} = (-1)^{4-i}\binom{4}{3}\frac{D^{4-i}}{d} \qquad \text{i=1,2,3.}$$

(5.10)
$$\left|A_1\right| = \begin{vmatrix} 1 & -4\frac{D}{d} & 6\frac{D^2}{d} & -4\frac{D^3}{d} \\ 0 & \frac{m+D^4}{d} & -4\frac{D}{d} & 6\frac{D^2}{d} \\ 0 & -4\frac{D^3}{d} & \frac{m+D}{d} & -4\frac{D}{d} \\ 0 & 6\frac{D}{d} & -4\frac{D^3}{d} & \frac{m+D^4}{d} \end{vmatrix}$$

$$ms_{3,1} = (-1)^{4-3}\binom{4}{1}\frac{D^{4-3}}{d} = -4\frac{D}{d}$$

$$ms_{2,1} = (-1)^{4-2}\binom{4}{2}\frac{D^{4-2}}{d} = 6\frac{D^2}{d}$$

$$ms_{1,1} = (-1)^{4-1}\binom{4}{3}\frac{D^{4-1}}{d} = -4\frac{D^3}{d}$$

$$ms_{0,1} = \frac{m + (-1)^4 D^4}{d} = \frac{m + D^4}{d}$$

Solving (5.10) we get

$$\frac{m}{d}\left|A_1\right| = \frac{m}{d}\left[\left(\frac{m+D^4}{d}\right)^3 + \frac{28D^4m^2 + 28D^8m}{d^3}\right]$$

Let n=4,D=4, d=2 then $m = D^4 + d = 258$ and $\sum_{j=0}^{\infty}\binom{4j+3}{4j}\left(\frac{128}{129}\right)^j = 135794945.$

In conclusion (5.7) and (5.9) provide us more classes of series and their sums since we can also solve for each of the $t_{i,1}$ (i=1,..,n -1).

For example if i=n-2 we have

(5.11) $$t_{n-2,1} = \frac{dD^2}{m^2}\sum_{j=0}^{\infty}\binom{(j+1)n+1}{jn+2}\left(\frac{D^n}{m}\right)^j$$

(5.12) $$\text{and } t_{n-2,1} = \begin{vmatrix} s_{0,1} & ms_{n-1,1} \ldots 1 & ms_{1,1} \\ s_{1,1} & s_{0,1} \ldots 0 & ms_{2,1} \\ \ldots & \ldots \ldots\ldots & \ldots \\ s_{n-1,1} & s_{n-2,1} \ldots 0 & s_{0,1} \end{vmatrix}$$

where $s_{i,1}$ and $s_{0,1}$ are as in (5.2)

(5.13) $$\text{or } \sum_{j=0}^{\infty}\binom{(j+1)n-i+1}{jn-i}\frac{D^{jn-1}}{m^j} = \left(\frac{m}{d}\right)\left|A_{i+1}\right|$$

as (5.3) where $s_{i,1}$ and $s_{0,1}$ are as in (5.2).

Section 6. Some infinite series and their sums from (BGEA)

In this section the author use a technique to find sums of some infinite series from units in algebraic number fields. Starting with a unit $e = \frac{(w-D)^n}{d}$ from (BGEA) we compute e^k and e^{-k} powers of this units in Q(w), $w = \sqrt[n]{D^k + d}$, D∈N, d∈Z, and d | D and we will evaluate the sums of some infinite series.

We consider a units $e = \frac{(w-D)^n}{d}$ from (BGEA) and use it to evaluate the sums of some infinite series.

THEOREM 1. Let $e=(w-D)^n$ in Q(w); $w=\sqrt[n]{D^n+d}$, D∈N, d∈Z, d | D, k,n∈N, n>2, k>1 and $m=D^n+d$.

(6.1)
$$A=\begin{bmatrix} s_{0,k} & ms_{n-1,k} & \dots & ms_{2,k} & ms_{1,k} \\ s_{1,k} & s_{0,k} & \dots & ms_{3,k} & ms_{2,k} \\ \dots & \dots & \dots & \dots & \dots \\ s_{n-2,k} & s_{n-3,k} & \dots & s_{0,k} & ms_{n-1,k} \\ s_{n-1,k} & s_{n-2,k} & \dots & s_{1,k} & s_{o,k} \end{bmatrix}$$

where

(6.2)
$$\begin{cases} s_{0,k}=\frac{1}{d^k}(m^k+(-1)^n\binom{kn}{n}D^n m^{k-1}+(-1)^{2n}\binom{kn}{2n}D^{2n}m^{k-2} \\ +\dots+(-1)^{kn}\binom{kn}{kn}D^{kn}) \\ s_{i,k}=\frac{1}{d^k}((-1)^{n-i}\binom{kn}{n-i}D^{n-i}m^{k-1}+(-1)^{2n-i}\binom{kn}{2n-i}D^{2n-i}m^{k-2}+ \\ +\dots+(-1)^{kn-i}\binom{kn}{kn-i}D^{kn-i}),\ i=1,\dots,n-1 \end{cases}$$

Then

(6.3)
$$\begin{cases} \sum_{j=0}^{\infty}\binom{(j+k)n-1}{jn}\frac{D^{jn}}{m^j}=\left(\frac{m}{d}\right)^k|A_1| \\ \sum_{j=1}^{\infty}\binom{(j+k)n-i-1}{jn-1}\frac{D^{jn-i}}{m^j}=\left(\frac{m}{d}\right)^k|A_{i+1}|,i=1,\dots,n-1 \end{cases}$$

where $|A_1|$ is $|A|$ with the first column replaced by the vector (1,0,…,0); $|A_{i+1}|$ is $|A|$ with the i-th column replaced by the vector (1,0,…,0).

Proof : Suppose

(6.4)
$$\begin{cases} e^k=s_{0,k}+s_{1,k}w+\dots+s_{n-1,k}w^{n-1},\ (k=1,2\dots) \\ e^{-k}=t_{0,k}+t_{1,k}w+\dots+t_{n-1,k}w^{n-1}, \end{cases}$$

We perform $1=e^k e^{-k}$ reducing the powers of w, knowing that $w^n=D^n+d=m$ and obtain the system of n equations:

(6.5)

$$\begin{cases} 1 = s_{0,k}\cdot t_{o,k} + m(s_{n-1,k}\cdot t_{1,k} + s_{n-2}\cdot t_{2,k} + \ldots + s_{1,k}\cdot t_{n-1,k}) \\ 0 = (s_{1,k}\cdot t_{o,k} + s_{0,k}\cdot t_{1,k}) + m(s_{n-1,k}\cdot t_{0,k} + s_{n-2,k}\cdot t_{2,k} + \ldots + s_{2,k}\cdot t_{n-1,k}) \\ 0 = (s_{2,k}\cdot t_{0,k} + s_{1,k}\cdot t_{1,k} + s_{0,k}\cdot t_{2,k}) + m(s_{n-1,k}\cdot t_{3,k} + s_{n-2,k}\cdot t_{4,k} + \ldots + s_{3,k}\cdot t_{n-1,k}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 = (s_{n-2,k}\cdot t_{0,k} + s_{n-3,k}\cdot t_{1,k} + \ldots + s_{o,k}\cdot t_{n-2,k}) + m(s_{n-1,k}\cdot t_{n-1,k}) \\ 0 = s_{n-1,k}\cdot t_{0,k} + s_{n--2,k}\cdot t_{1,k} + \ldots + s_{0,k}\cdot t_{n-1,k} \end{cases}$$

Taking the $t_{i,k}$' s as unknowns, the determinant of the system is $|A|$, where A is as in (6.1).

Since $w^n = m$, Q(w) has basis $1, w, \ldots w^{n-1}$, so that any algebraic number in Q(w) has the form $\sigma = x_1 + x_2 w + \ldots + x_n w^{n-1}$, $x_i \in Q$ (i=1,..,n -1). The norm of ó is a polynomial in the x_i and N(ó)=±1. So we have $(-1)^{n-1} S_n = m$, S_j=0 for j=n, ó=e and S_n and S_j are the elementary symmetric functions in $D_0, D_1, \ldots D_n$ and

$$(-1)^n S_n = (-1)^n D_0 D_1 \ldots D_{n-1} = d.$$

In this case

(6.6)
$$N(e^k) = \begin{bmatrix} s_{0,k} & s_{1,k} & \ldots & s_{n-2,k} & s_{n-1,k} \\ ms_{n-1,k} & s_{0,k} & \ldots & s_{n-3,k} & s_{n-2,k} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ms_{2,k} & ms_{3,k} & \ldots & s_{0,k} & s_{1,k} \\ ms_{1,k} & ms_{2,k} & \ldots & ms_{n-1,k} & s_{0,k} \end{bmatrix}$$

which is the transpose of the $|A|$ with A in (6.1).

Also, $$N(e^k) = N\left[\frac{(w-D)^{kn}}{d^k}\right] = \left[N\left(\frac{w-D^k}{d^k}\right)\right]^n$$

and

$$N(w-D) = \begin{vmatrix} -D & 1 & 0\ldots & 0 & 0 \\ 0 & -D & 1\ldots & 0 & 0 \\ \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ 0 & 0 & 0\ldots & -D & 1 \\ m & 0 & 0\ldots & 0 & -D \end{vmatrix} = (-D)(-1)^{n-1} D^{n-1} + (-1)^{n-1} m$$

and N(w-D)=N(e)=1.

Since N(w-D)=1. Then $(N-D)^k=1$.

Now we apply Cramer' s rule to solve for $t_{0,k}$ in (6.5) and

(6.7)
$$t_{0,k}=\begin{bmatrix}1 & ms_{n-1,k} & \dots ms_{2,k} & ms_{1,k}\\ 0 & s_{o,k} & \dots ms_{3,k} & ms_{2,k}\\ \dots & \dots & \dots\ \dots & \dots\\ 0 & s_{n-3,k} & \dots\ s_{0,k} & ms_{n-1,k}\\ 0 & s_{n-2,k} & \dots\ s_{1,k} & s_{o,k}\end{bmatrix}$$

and

$$e^k=\frac{(w-D)^{kn}}{d^k}=\frac{1}{d^k}\left(w^{kn}-\binom{kn}{n}w^{nk-1}D+\dots+(-1)\binom{kn}{n}w^{kn-n}D^n+\dots\right.$$
$$\left.+(-1)^{2n}\binom{kn}{2n}w^{kn-2n}D^{2n}+\dots+(-1)^{kn}\binom{kn}{kn}D^{kn}\right)$$

and since w^n=m.

$$s_{0,k}=\frac{1}{d^k}\left(m^k-(-1)^n\binom{kn}{n}D^n m^{k-1}+(-1)^{2n}\binom{kn}{2n}D^{2n}w^{k-2}+\dots+(-1)^{kn}\binom{kn}{kn}D^{kn}\right)$$

$$s_{i,k}=\frac{1}{d^k}\left((-1)^{n-1}\binom{kn}{n-i}D^{n-i}m^{k-1}+(-1)^{2n-i}\binom{kn}{2n-i}D^{2n-i}w^{k-2}+\dots+(-1)^{kn-i}\binom{kn}{kn-i}D^{kn-i}\right)$$

$i=1,\dots n-1$

(6.7) solves for $t_{0,k}$ in terms of the parametric forms for the (6.2). Another way to write $t_{0,k}$ is $e^{-k}=d^k(w-D)^{-nk}=\left(\frac{d}{m}\right)^k\left(1-\frac{D}{w}\right)^{-nk}$ and since $\left|\frac{D}{w}\right|<1$ we have

(6.8)
$$e^{-k}=\left(\frac{d}{m}\right)^k\sum_{j=0}^{\infty}\binom{jn+kn-1}{jn}\frac{D^{jn}}{m^j}+\left(\frac{d^k}{m^k}\sum_{j=0}^{\infty}\binom{jn+(k+1)n-2}{jn+n-1}\frac{D^{jn+n-1}}{m^{j+1}}\right)w+$$
$$+\dots+\left(\frac{d^k}{m^k}\sum_{j=0}^{\infty}\binom{jn+kn+1}{jn+2}\frac{D^{jn+2}}{m^{j+1}}\right)w^{n-2}+\left(\frac{d^k}{m^k}\sum_{j=0}^{\infty}\binom{jn+kn}{jn+1}\frac{D^{jn+1}}{m^{j+1}}\right)w^{n-1}.$$

In particular from (6.8) we have

$$t_{o,k}=\left(\frac{d}{m}\right)^k\sum_{j=0}^{\infty}\binom{jn+kn-1}{jn}\left(\frac{D^n}{m}\right)^j$$

or

$$(6.9)\quad \begin{cases} t_{o,k} = \left(\dfrac{d}{m}\right)^k \displaystyle\sum_{j=0}^{\infty} \binom{(j+k)n-1}{jn} \left(\dfrac{D^n}{m}\right)^j \\ t_{i,k} = \left(\dfrac{d}{m}\right)^k \displaystyle\sum_{j=0}^{\infty} \binom{(j+k)n-i-1}{jn-i} \dfrac{D^{jn-1}}{m^j}, i = 1,\dots,n-1. \end{cases}$$

(6.9) gives a parametric family of infinite series and an explicit form of sums is given by (6.7) as (6.3) with $s_{0,k}$ and $s_{i,k}$ as in (6.2).
Example for n=5 and k=1, $m=w^4=D^4+d$ and $d \mid D$.

$$(6.10)\quad \begin{aligned} \frac{m}{d}\left|A_1\right| &= \sum_{j=0}^{\infty} \binom{5j+4}{5j} \left(\frac{D^5}{m}\right)^j = \\ &= \left(\frac{m}{d}\right)\left(1 + \frac{375D^5 m}{d^2} + \frac{1500D^{10}m - 1500D^5 m^2}{d^3} + \frac{1250D^{15}m - 1875D^{10}m^2 + 1250D^5 m}{d^4}\right) \end{aligned}$$

Since $\left|A_1\right| = \begin{vmatrix} 1 & ms_{4,1} & ms_{3,1} & ms_{2,1} & ms_{1,1} \\ 0 & s_{0,1} & ms_{4,1} & ms_{3,1} & ms_{2,1} \\ 0 & s_{1,1} & s_{0,1} & ms_{4,1} & ms_{3,1} \\ 0 & s_{2,1} & s_{1,1} & s_{0,1} & ms_{4,1} \\ 0 & s_{3,1} & s_{2,1} & s_{1,1} & s_{0,1} \end{vmatrix}$

$$s_{0,1} = \frac{m + (-1)^5 D^5}{d} \text{ and}$$

$$s_{i,1} = (-1)^{5-i}\binom{5}{5-i}\frac{D^{5-i}}{d}, i = 1,2,3,4$$

$$s_{0,1} = \frac{m - D^5}{d}; s_{2,1} = -10\frac{D^3}{d}; s_{4,1} = -5\frac{D}{d};$$

$$s_{1,1} = 5\frac{D^4}{d}; s_{3,1} = 10\frac{D^2}{d}$$

and

$$|A_1| = \begin{vmatrix} 1 & -5m\frac{D}{d} & 10m\frac{D^2}{d} & -10\frac{D^3}{d} & 5m\frac{D^4}{d} \\ 0 & \frac{m-D^5}{d} & -5m\frac{D}{d} & 10m\frac{D^2}{d} & -10m\frac{D^3}{d} \\ 0 & 5\frac{D^4}{d} & \frac{m-D^5}{d} & -4\frac{D}{d} & 10m\frac{D^2}{d} \\ 0 & -10\frac{D^3}{d} & 5\frac{D^4}{d} & \frac{m-D^5}{d} & -5m\frac{D}{d} \\ 0 & 10\frac{D^2}{d} & -10\frac{D^3}{d} & 5\frac{D^4}{d} & \frac{m-D^5}{d} \end{vmatrix}$$

or

$$|A_1| = 1 + \frac{375D^5m}{d^2} + \frac{1500D^{10}m - 1500D^5m^2}{d^3} + \frac{1250D^{15}m - 1875D^{10}m^2 + 1250D^5m^3}{d^4}$$

which implies (6.10).

For numerical $d, D \in N$ and $d \mid D$ we will get a number for the sum in (6.10).

Section 7. (BGEA) the explicite solution of HILBERT 10-th problem

Hilbert' s dream and advice (zahlbericht) of tailoring a universal always periodic algorithm by means of which we can solve all the open questions in algebraic number theory of n-dimension, questions which were completely solved in quadratics from the always periodicity of the Euclidean Algorithm (EA, is known as Hilbert' s 10-th problem. In other words Hilbert asked for the General Euclidean Algorithm and its proof of always (unrestricted) periodicity by an equivalent n-dimensional Euler Lagrange theorem from quadratics.

Logicians (without giving an explicit proof) proved Hilbert' s 10-th problem by logic showing that a such algorithm to be periodic always does not exist.

Asking for this universal algorithm Hilbert asked for the Euler System (ES) of the algebraic number theory which is the (algebra) number theory of the n-dimensional Euclidean geometry (E^n) exactly as (EA) is the (ES) of the algebraic number theory in quadratics which is the (algebra) or number theory of 2-dimensional Euclidean geometry.

There were some attempts to prove explicitly Hilbert' s 10th problem by polynomial representation but this leads nowhere in proving all the n-dimensional open problems in the algebraic number theory. The answer to that is that this explicit polynomial representation is not an algorithmic explicit representation. In all of the previous sections of this book we proved all of the open problems in higher dimensions showing that (BGEA) is a very powerful algorithm when it is periodic. All o these problems in higher dimensions do not have solutions when (BGEA) fails to be periodic and this is when $d \mid D$. (BGEA) is the evolutionary development of Jacobi, Perron, Hasse-Bernstein and Baica algorithms. (BGEA) solved up to its restricted periodicity all the open questions in the algebraic number theory, and this proves that (BGEA) is the explicit form of Hilbert' s demanded algorithm.

In Hilbert' s 10th problem, he was asking for the General Euclidean Algorithm (GEA) which will prove from its periodicity all the open questions in n-dimensions which were proved in quadratics from the periodicity of the (EA). In this book we showed that (BGEA) does this when $d \mid D$. With this we proved that (EA) is the Euler System (ES) in quadratics and (BGEA) is the Euler System (ES) in n-dimensions for the algebraic number theory which is the number theory of the n-dimensional Euclidean geometry (E^n).

Section 8. Conclusions

As it can be seen, it is not only the beginning Euclidean algorithm (EA) and the end Baica' s general Euclidean algorithm (BGEA) it is so much else in between than only the gap that makes them apart for more than 2000 years. It put together the work of great mathematicians during the entire history of mathematics beginning with Euclid and finishing with Baica.

This so much else in between is all of the genial work of these great mathematicians before me, who historically paved the way for me to finish this final step and give the mathematics this very powerful tool which is the General Euclidean Algorithm. Also this so much in between helped me to produce (BGEA) the General Euclidean Algorithm which is the Euler System (ES) in E^n.

All of those great mathematicians aimed to produce the General Euclidean Algorithm and to prove its periodicity some time in their life, and with their genial work which I put together, their dreams become a realisation and now we have (BGEA) to be this General Euclidean Algorithm. The (BGEA) will dominate mathematics for higher dimension fields over the years to come, exactly as (EA) dominated mathematics for quadratic fields for so many years in the past.