CHAPTER 1

THE RIEMANN HYPOTHESIS

by Aldo Peretti

1.0. INTRODUCTION

In 1859, G.F.B.Riemann published a most famous paper concerning the distribution of prime numbers, with the title: "On the quantity of prime numbers below a given quantity", where, for the first time the methods of complex variable functions were used in order to determine $\pi(x)$: the number of prime numbers $\leq x$.

His starting formula was the product decomposition that Euler had found for the zeta function

(1.0.1) $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

i.e. the formula

(1.0.2) $$\zeta(s) = \prod_{p} \frac{1}{1-1/p^s}$$

where p stands for the prime numbers. (Riemann used the letter s to denote the variable, $s = \sigma + it$; and this way of notation was unanimously used after him)

In the first part of the memoir, he proves the functional equation of the zeta function, and after this he deduces the formula

(1.0.3) $$\frac{\log \zeta(s)}{s} = \int_0^{\infty} \frac{f(x)}{x^{s+1}} dx$$

valid for $\sigma > 1$, and where

(1.0.4) $$f(x) = \pi(x) + 1/2\,\pi\left(\sqrt{x}\right) + 1/3\,\pi\left(\sqrt[3]{x}\right) + \ldots$$

This formula had been obtained formerly in 1848 by Tchebychev (whose work on the subject very likely was known to Riemann).

But he was unable to make the inversion of this formula, that Riemann succeeded to do, obtaining thus:

(1.0.5) $$f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\log \zeta(s)}{s} x^s \, ds \qquad (a > 1)$$

The remaining part of Riemann's paper is very obscure and confusing because of its excessive brevity. Fortunately some years ago E.C. Edwards published a wonderful book (ref. [12]) explaining and justifying step by step Riemann's reasoning.

This required about 200 pages, and put in evidence that he performed six hypotheses before arriving at his final formula:

1) There are infinitely many zeros of $\zeta(s)$ in the "critical strip" $0 \le \sigma \le 1$.

2) The quantity N(T) of these zeros in the rectangle $0 \le \sigma \le 1 \quad 0 < t < T$ is

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1+\log 2\pi}{2\pi} T + O\left(\frac{1}{T}\right)$$

3) The series $\sum |\rho|^{-2}$ is convergent, but $\sum |\rho|^{-1}$ diverges

4) The entire function

$$\xi(s) = ½ \pi^{-s/2} s(s-1) \zeta(s) \Gamma(s/2)$$

admits the product decomposition

$$\xi(s) = a e^{bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$

5) All the imaginary zeros ρ have real part 1/2

6) Let

$$\pi^*(x) = \sum_{2 \le n \le x} \frac{\Lambda(n)}{\log n} \qquad \pi_0(x) = \frac{1}{2}\left\{\pi^*(x+0) + \pi^*(x-0)\right\}$$

Then holds that

$$\pi_0(x) = \pi(x) + 1/2\, \pi\left(\sqrt{x}\right) + 1/3\, \pi\left(\sqrt[3]{x}\right) + \ldots$$

$$= li(x) - \sum_{\rho} li\left(x^{\rho}\right) + \int_x^{\infty} \frac{du}{(u^2-1)u.\log u} - \log 2$$

where li(x) denotes the logarithmic integral function; $\pi(x)$ is the number of primes $\le$ x , and ρ denotes the imaginary zeros of the zeta function.

The hypothesis 1), 3) and 4) were proved in 1892 and 1893 by Hadamard. The hypothesis 2), with error term 0(log T) was proved in 1894 by von Mangoldt, who also proved hypothesis 6) (but he used an alternative way).

There is a numerical and irrelevant mistake in the formula for $\pi_0(x)$, where Riemann wrote $\xi(½)$ instead of -log 2.

Hence at present remains unproved hypothesis 5) only.

Notice that Riemann's formula for N(T) indeed gives the quantity of Gram points for $t \leq T$, up to a difference of $\pi/8$.

1.1. HOW THE HYPOTHESIS WAS NOT PROVED

Variant A)

It is mentioned in § 10.1, p.213-214 of Titchmarsh's textbook. (ref [56]). We have:

$$\xi\,(1/2 + it) = 2\int_0^\infty \Phi(u)\ \cos ut\ dt \tag{1.1.1}$$

WHERE

$$\Phi\,(u) = 2\sum_{n=1}^{\infty}\left(2n^4\pi^2 e^{\frac{9}{2}u} - 3n^2\,\pi e^{\frac{5}{2}u}\right) e^{-n^2\pi e^{2u}} \tag{1.1.2}$$

This series converges very rapidly and one might suppose that an approximation to the truth could be obtained by replacing it by their first terms.

The author has performed the exact calculation, and he proved that we are thus led to the formula

$$\xi(s) = \frac{1}{2} + \frac{s(s-1)}{2}\left\{\pi^{-s/2}\left(\sum_{n-1}^{N}\frac{\Gamma_{n^2x}}{n^s} + 0\left(\frac{e^{-\frac{x}{4}t}}{t^{\frac{3}{2}}}\right)\right) + \right. \tag{1.1.3}$$

$$\left. +\ \pi^{-\frac{1-s}{2}}\left(\sum\frac{\Gamma_{n^2x}\left(1-\frac{s}{2}\right)}{n^{1-s}} + 0\left(\frac{e^{-\frac{x}{4}t}}{t^{\frac{3}{2}}}\right)\right)\right\}$$

where $N = \left[\frac{\sqrt{t}}{2}\right]$ and $\Gamma_{n^2\pi}(x)$ denotes the incomplete gamma function. See ref. [30].

It is evident now the slowness of convergence of both series at right, because are necessary $0(\sqrt{t})$ terms in order to obtain a satisfactory accuracy. Hence, it has not any special advantage over the use of the Riemann – Siegel formula.

Variant B)

In other place of his book ref.[56] (Chapter III § 3,1 p.38) Titchmarsh states that "The problem of the zero-free region" (of the zeta function) "appears to be a question of extending the sphere of influence of the Euler product beyond its actual region of convergence, In fact, the deepest theorems on the distribution of the zeros of $\zeta(s)$ are obtained in the way suggested.

But the problem of extending the sphere of influence of (the Euler product) ... to the left of $\sigma = 1$ in any effective way appears to be of extreme difficulty"

The "extremely difficult problem" was solved in 1991 by the author (ref [34]), who proved the product formula

$$\zeta(s) = \prod_{p \le x} \frac{1}{1-\frac{1}{p^s}} \frac{e^{\sum Li_1(x^{p-s})}}{e^{Li_1(x^{1-s})}} e^{\sum Li_1(x^{-2-s})} \tag{1.1.4}$$

for integral positive $x \geq 2$ and every s.

Very unfortunately, this does not give us any information concerning the zeros.

In change, in p.50 of the same reference, the author proved that there are infinitely many natural numbers x such that if

$$\theta = 1 - \frac{2 \log \log x}{\log x} + 0\left(\frac{1}{\log x}\right) \tag{1.1.5}$$

there are not zeros for $\sigma > \theta$

1.2. HOW THE HYPOTHESIS WAS PROVED

In the week from sept.7 to sept. 14/2002 it was held in Falda del Carmen (in the argentine province of Córdoba) the "XI Taller de Ciencias Básicas Espaciales" organized by the United Nations and the European Spatial Agency.

There the pakistan mathematician M. Aslam Chaudhry, professor of the King Fahd University of Petroleum and Minerals of Saudi Arabia exposed a proof of the Riemann hypothesis that was found correct by the audience. Prof. Chaudhry did not deliver any copy of his contribution, nor gave any abstract of it. His electronic address is Maslam @ Kfupm.edu.sa.

In the Nº 10, January 2001 issue of the "Italian Journal of Pure and Applied Mathematics" was published a paper of the author entitled "The functions N(T) and N_0 (T) of the Riemann zeta function", (ref.[39]).

The contents of the first two pages of the paper can be summarized in a few lines as follows:

The function

$$\xi (s) = 1/2\, s\, (s-1)\, \pi^{-s/2}\, \Gamma (s/2)\, \zeta (s)$$

is real for $s = 1/2 + it$ and real t, as proved by Riemann. Hence, the same thing is true for

(1.2.1) $$F (s) = \pi^{-s/2}\, \Gamma (s/2)\, \zeta (s)$$

Equating arguments of both sides in (1.2.1.) we obtain:

(1.2.2.) $$\arg F(1/2 + it) = -t/2 \log \pi + \arg \Gamma(1/4 + it/2) + \arg \zeta(1/2 + it) \pm m\pi$$

As F (1/2 + it) is a real quantity, must hold that

(1.2.3.) $$\arg F (1/2 + it) = \pm \pi \pm 2k\pi$$

Replacing in (1.2.2.) we obtain:

(1.2.4.) $$\pm \pi \pm 2k\pi = - t/2 \log \;\; + \arg \Gamma (1/4 + it/2) + \arg \zeta (1/2 + it) \pm m \pi$$

which is clearly equivalent to:

(1.2.5.) $$\pm n \pi = - t/2 \log \pi + \arg \Gamma (1/4 + it/2) + \arg \zeta (1/2 + it)$$

This equation must hold identically for every real t.

The preceding formula admits a double interpretation:

A) It gives the quantity of zeros $N_0(T)$ in the critical line $\sigma = ½$ between ordinates t=0 and t = ± T

B) It enables us to evaluate arg ζ (1/2 + it)

The validity of (1.2.5) can be checked in a variety of ways.

1) By use of the Haselgrove-Miller tables (ref. [15]). This can be accomplished in two different ways. On the one hand the tables give separately the values of R ζ (½ + it) and I ζ (½ + it), so that one can evaluate.

$$\arg \zeta (1/2 + it) = \text{arc tg} \frac{I\zeta(1/2+it)}{R\zeta(1/2+it)} \pm m\pi$$

As ζ(1/2) = - 1,460355, it seems a correct conventional thing to adopt above that

$$\arg \zeta (1/2) = \pi$$

This is the initial value used to draw the graph of arg ζ (½ + it) given in fig.1.

On the other hand, the tables tabulate the values of θ = arg Γ (¼ + it/2) and thus provide an alternative way to calculate arg ζ (½ + it) thanks to (1.2.5).

The numerical details of this checks were given in ref. [33] and the double agreement of the tables with (1.2.5.) is perfect.

2) Formula (1.2.5) can be derived from the Cauchy-Riemann equations for analytic functions. See ref [33]

3) It can be derived from the argument “principle” when applied to ξ (s). See § 1.3.7 below.

4) It can be derived from the functional equation of the zeta function. See § 1.3.4 below.

5) It can be derived from the approximate functional equation of the zeta function. See § 1.3.5 below.

6) It can be derived from an observation made by Titchmarsh. See § 1.3.6 below.

1.3. COMPARISON WITH THE CONTOUR INTEGRATION

Once we feel fairly sure by these six checks about the validity of (1.2.5) we compare it with the result of the contour integration performed by Stieltjes, von Mangoldt and Backlund in order to determine N(T), the quantity of zeros in the critical strip between ordinates t = 0 and t = T.

In ref [56], Chapter IX, § 9.3, line 13 p.179 we find the following.:

Theorem 1.3.1. - The positive ordinates of the zeros of the zeta function in the critical strip are defined by the condition

(1.3.1) $$n\pi = \Delta \arg s(s-1) + \Delta \arg \pi^{-s/2} + \Delta \arg \Gamma(s/2) + \Delta \arg \zeta(s) =$$
$$= \pi - t/2 \log \pi + \arg \Gamma(1/4 + it/2) + \arg \zeta(1/2 + it)$$

Stated in a slightly different form, there are zeros in the critical strip every time that

(1.3.2) $$\pm m\pi = - t/2 \log \pi + \arg \Gamma(1/4 + it/2) + \arg \zeta(1/2 + it)$$

where we have replaced n -1 by m.

But (1.3.2) and (1.2.6) coincide for every t. Hence

$$N(t) = N_0(t)$$

which is the Riemann hypothesis.

As a further check, Prof. Gerd Faltings, after having read ref [39] said he had been unable to find any mistake in it. (except the typographical ones, that very regrettably were numerous).

1.4. HERE WE DEVELOP ALTERNATIVE 4), MENTIONED ABOVE

In the functional equation of the zeta function, which we write as:

$$\zeta(s)\ \Gamma(s/2) = \pi^{s-½}\ \Gamma((1-s)/2)\ \zeta(1-s)$$

we choose $s = ½ + it$, and equate the arguments of both members. We obtain:

(1.4.1.) $$\arg\zeta(½ + it) + \arg\Gamma(¼ + i\,t/2) =$$

$$= t\log\pi + \arg\Gamma(¼ - i\,t/2) + \arg\zeta(½ - it) \pm 2n\pi$$

But

$$\arg\zeta(½ + it) = -\arg\zeta(½ - it)$$

$$\arg\Gamma(¼ - i\,t/2) = -\arg\Gamma(¼ - i\,t/2)$$

Replacing these values in (1.4.1) we obtain:

(1.4.2) $$2\arg\zeta(½ + it) + 2\arg\Gamma(¼ + it/2) = t\log\pi \pm 2n\pi$$

which is again (1.2.5)

1.5. HERE WE DEVELOP ALTERNATIVE 5) MENTIONED EARLIER ABOVE

As known (see for instance ref (2)), the approximate functional equation of the zeta function can be written (when we choose $s = (½ + it)$) as:

(1.5.1) $$\zeta(½ + it) = = \sum_{1}^{m}\frac{1}{n^{1/2+it}} + e^{-i2\theta}\sum_{1}^{m}\frac{1}{n^{1/2-it}} + e^{-i\theta}R_m$$

where:

$$m = \left[\sqrt{\frac{t}{2\pi}}\right] \quad Rm = (-1)^m\left(\frac{t}{2\pi}\right)^{-1/4}\sum_{r=0}^{\infty}\left(\frac{t}{2\pi}\right)^{-r/2}\phi_r\left(2\sqrt{\frac{t}{2\pi}} - 2\left[\sqrt{\frac{t}{2\pi}}\right] - 1\right)$$

$$\theta = -\ t/2\ \log\pi + \arg\Gamma(¼ + it/2)$$

The form of the functions Φ_r can be consulted in ref. [56], but it is irrelevant for what follows.

We remark now that $e^{-i\theta}$ Rm can be written as :

$$e^{-i\theta}\ Rm = M\,e^{i\alpha m} + e^{-i2\theta}\,M\,e^{-i\alpha m}$$

In fact, this is the same that

$$Rm = Me^{i(\theta + \alpha m)} + M e^{-i(\theta + \alpha m)} = 2M \cos(\theta + \alpha_m)$$

and simply:

$$M = \left|\frac{Rm}{2}\right| \quad \text{Cos}(\theta + \alpha_m) = \frac{Rm}{|Rm|}$$

These equations determine both M and αm. Hence we can write that

(1.5.2) $$\zeta(1/2 + it) = \sum_{1}^{m} \frac{1}{n^{1/2+it}} + M e^{i\alpha_m} + e^{-2i\theta}\left\{\sum_{1}^{m} \frac{1}{n^{1/2-it}} + M e^{-i\alpha_m}\right\}$$

and the zeros of $\zeta(½ + it)$ are given by the formula

(1.5.3) $$0 = \sum_{1}^{m} \frac{1}{n^{1/2+it}} + M e^{i\alpha_m} + e^{-2i\theta}\left\{\sum_{1}^{m} \frac{1}{n^{1/2+it}} + M e^{-i\alpha_m}\right\}$$

If $t \neq t_\nu$, with t_ν = a zero of the right bracket, we can divide both numbers by this bracket thus obtaining:

(1.5.4) $$e^{-2i\theta} = \frac{\sum_{1}^{m} \frac{1}{n^{1/2+it}} + M e^{i\alpha_m}}{\sum_{1}^{m} \frac{1}{n^{1/2-it}} + M e^{-i\alpha_m}}$$

We observe now that the quotient at right is formed by two conjugate complex numbers, so that its modulus is exactly one :

Let

(1.5.5) $$\beta = \arg\left\{\sum_{1}^{m} \frac{1}{n^{1/2+it}} + M e^{i\alpha_m}\right\}$$

Then the above quotient has argument 2 β, and the former equation adopts this simple form:

$$e^{-2i\theta} = -e^{2i\beta}$$

the roots of which are

(1.5.6) $$2(\theta + \beta) = \pm \pi \pm 2k\pi$$

Now we evaluate β in terms of $\alpha_\zeta = \arg \zeta(½ + it)$. (1.5.2) can be written as

(1.5.7) $$\zeta(½ + it) = M_1 e^{i\beta} + e^{-i2\theta} M_1 e^{-i\beta} = M_1 \left\{e^{i\beta} + e^{-i(2\theta+\beta)}\right\}$$

where

$$M_1 = \left|\sum_{1}^{m} \frac{1}{n^{1/2+it}} + M e^{i\alpha_m}\right|$$

Let us put :

(1.5.8) $$\zeta\,(½ + it) = M_\zeta\, e^{\,i\alpha\zeta}$$

Replacing in (1.5.3) we get:

$$M_\zeta\, e^{\,i\alpha_\zeta} = M_1 \left\{ e^{i\beta} + e^{-i\,(2\theta+\beta)} \right\}$$

and

$$\frac{M_\varsigma}{M_1}\; e^{i(\alpha_\varsigma-\beta)} = 1 + e^{-i(2\theta\,+\,2\beta)}$$

Taking logarithms:

$$\mathrm{Log}\,\frac{M_\zeta}{M_1} + i\left(\alpha_\zeta - \beta\right) = \log\left|1+e^{i2(\theta+\beta)}\right| - i\left(\mathrm{arc.tg.}\,\frac{\sin 2\,(\theta+\beta)}{1+\cos 2\,(\theta+\beta)} \pm n\pi\right)$$

Equating imaginary parts :

(1.5.9) $$\alpha_\zeta - \beta = -\,\mathrm{arc\,tg}\,\frac{\sin 2\,(\theta+\beta)}{1+\cos 2\,(\theta+\beta)} \pm n\pi$$

But according to (1.5.3)

$$\sin 2\,(\beta + \theta) = \sin\,(\pm\pi) = 0$$

$$\cos 2\,(\beta + \theta) = \cos\,(\pm\pi) = -1$$

so that the quotient

$$\frac{\sin 2(\theta+\beta)}{1+\cos 2(\theta+\beta)}$$

is of the type 0/0. Once the indetermination is saved by L'Hopital rule, we find $-\infty$ as its true value. Hence we deduce

(1.5.10) $$\mathrm{arc\ tg.}\,\frac{\sin 2\,(\vartheta+\beta)}{1+\cos 2\,(\vartheta+\beta)} = \pm\ \pi/2\ \pm\ r\pi$$

Thus (1.5.9) transforms itself to

(1.5.11) $$\alpha_\zeta - \beta = \pm\ \pi/2\ \pm\ m\pi$$

Replacing this value of β in (1.5.6), we obtain that there are zeros on the critical line every time that

(1.5.12) $$2\,(\theta + \alpha_\zeta) = \pm\, 2k\,\pi$$

or

$$\theta + \alpha_\zeta = \pm\, k\pi$$

which is again (1.2.5).

1.6. TITCHMARSH'S OBSERVATION

In p.181 § 9,4 of ref. [56], he states:

"The behavior of the function S (t) = $1/\pi$ arg ζ (½ + it)) appears to be very complicated. It must have a discontinuity k where t passes through the ordinate of a zero of ζ (s) of order k .... Between the zeros, N(t) is constant, so that the variation of S(t) must just neutralize that of the other terms" (in the expression of N(t)).

A glance at fig.1 shows that the behaviour of π S(t) is very simple: it consists of the curve t/2 log π - arg Γ (¼ + it/2) broken at the points t = γ_i. But Titchmarsh did not dispose of Haselgrove's tables when writing his book.

The last statement is particularly interesting: between zeros, the variation of S(t) must neutralize that of the other terms in the formula of N(t).

That is to say, that according to (1.3.1), between consecutive zeros must hold that:

$$\Delta\, t/2 \log \pi - \Delta \arg \Gamma\ (1/4 + it/2) = \Delta \arg \zeta\, (1/2 + it)$$

or

$$(t_1 - t_2) \log \pi - \left\{ \arg \Gamma\left(1/4 + it_1/2\right) - \arg \Gamma\left(1/4 + it_2/2\right) \right\} =$$

$$= \arg \zeta\ \left(1/2 + it_1\right) - \arg \zeta\left(1/2 + it_2\right)$$

If we regard t_1 as a variable and t_2 = constant, then we have:

(1.6.1) $$t \log \pi - \arg \Gamma\ (1/4 + it/2) = \arg \zeta\, (1/2 + it) + \text{constant}$$

Due to the fact that arg ζ (1/2 + it) has jumps of π at each simple zero, it is easily seen that the constant must be $\pm$ m π.

But then (1.6.1) is nothing but (1.2.5)!

1.7. USE OF THE ARGUMENT PRINCIPLE

As known, the "argument principle" states that the quantity N of zeros if an analytic function f(s) inside a contour C where it has no poles, is equal to the variation of arg f(s) along C divided by 2π.

When applied to the function

$$\xi^*(s) = \pi^{-s/2}\, \Gamma(s/2)\ \zeta(s)$$

with functional equation

$$\xi^*(s) = \xi^*(1-s)$$

it gives

$$N = \frac{1}{2\pi} \Delta_c \arg \xi^* (s)$$

Next, we choose as contour C a thin strip of width δ surrounding the critical line between ordinates $t = \pm T$, and after we let $\delta \to 0$,. Then we obtain for the quantity N_0 of zeros along the critical line:

$$N_0 = \frac{1}{2\pi} \left\{ \arg \xi (1/2 + it) - \arg \xi (1/2 - it) \right\} = \frac{1}{\pi} \arg \xi (1/2 + it) =$$

$$= \frac{1}{\pi} \left\{ -t \log \pi + \arg \Gamma (1/4 + it/2) + \arg \zeta (1/2 + it) \right\}$$

which is again (1.2.5)

1.8. THE CASE OF THE L-SERIES

As known, the functional equation of the $L(s,\chi)$ functions is:

$$\xi(s,\chi) = \left(\frac{k}{\pi}\right)^{-(s+\delta)/2} \Gamma\left(\frac{s+\delta}{2}\right) L(s,\chi) =$$

(1.8.1) $$= g(\chi)\, i^{-\delta} k^{-1/2} \left(\frac{k}{\pi}\right)^{-\frac{(1-s+\delta)}{2}} \Gamma\left(\frac{1-s+\delta}{2}\right) L(1-s, \bar{\chi}) = \xi(1-s, \bar{\chi})$$

where: $$\delta = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases} \qquad \delta = \frac{1-\chi(-1)}{2}$$

$$g(\chi) = \sum_{a=1}^{k} \chi(a) \exp(2\pi i a / k)$$

and χ is a primitive character modulo k. (also called the conductor of χ). See, for instance, the book in ref. [18]. When we choose $s = 1/2 + it$ we obtain:

$$\left(\frac{k}{\pi}\right)^{1/4 + it/2 + \delta/2} \Gamma(1/4 + it/2 + \delta/2)\ L(1/2 + it, \chi) =$$

$$= g(\chi)\, i^{-\delta} k^{-1/2} \left(\frac{k}{\pi}\right)^{1/4 + it/2 + \delta/2} \Gamma(1/4 - it/2 + \delta/2)\quad L(1/2 - it, \bar{\chi})$$

Equating arguments of both sides we deduce:

$$t/2 \log \frac{k}{\pi} + \arg \Gamma\ (1/4 + \delta/2 + it/2) + \arg L(1/2 + it, \chi) =$$

$$= \arg g(\chi) - \delta\, \pi/2 \pm 2r\pi\delta - t/2 \log (k/\pi) + \arg \Gamma (1/4 - it/2 + \delta/2) + \arg L(1/2 - it, \bar{\chi}) \pm 2n\pi$$

or still

(1.8.2) $t \log k/\pi + 2\arg \Gamma(¼ + \delta/2 + it/2) + 2 \arg L(½ + it, \chi) =$

$$= -\delta \pi/2 + \arg g(\chi) \pm 2n\pi$$

because

$$\arg \Gamma(¼+\delta/2 - it/2) = -\arg \Gamma(¼+\delta/2 + it/2)$$

and

$$\arg L(½ - it, \bar{\chi}) = -\arg L(½ + it, \chi,)$$

due to the fact that

$$\xi(\bar{s}, \chi) = \overline{\xi(s, \bar{\chi})}$$

Hence, there are zeros in the critical line every time that

(1.8.3)

$$\pm n\pi = t/2 \log \frac{k}{\pi} + \arg \Gamma(¼ + \delta/2 + it/2) + \arg L(½ + it, \chi) + \arg g(\chi) + \delta\pi/4$$

On the other hand, according to ref. [44], the value of N_χ (T), the quantity of zeros of L (s,χ) in the rectangle $0 \le \sigma \le 1 \quad 0 \le t \le T$ for every χ mod k is:

(1.8.4) $$\pi N\chi(T) = \Delta C_1 \arg (k/\pi)^{(s+a)/2} + \Delta C_1 \arg \Gamma(s+\delta)/2 + \Delta C_1 \arg L(s,\chi)$$

where C_1 is composed of the two spans (5/2; 5/2 + iT) and (5/2 + iT , 1/2 + iT). (But if χ(-1) = -1 there is an extra zero)

It follows that there are zeros every time that

(1.8.5)

$$\pm n\pi = t/2 \log \frac{k}{\pi} + \arg \Gamma(1/4 + \delta/2 + it/2) + \arg L(1/2 + it, \chi) - \arg L(5/2, \chi) + \delta\pi$$

Comparing (1.8.3) with (1.8.5), we remark that the difference of their constant terms, i.e.:

(1.8.6) $$-\arg L(5/2, \chi) + \delta \pi - \arg g(\chi) - \delta \pi/4$$

is equal to the quantity of the real zeros of the L function in the interval $0 < \sigma < 1$. We know practically nothing about the quantities involved in (1.8.6), so that we are unable to draw any conclusion on this subject.

But when t >o it is evident that whenever the expression (1.8.5) increases by r π, the same thing occurs in expression (1.8.3) in every case.

We deduce thus that all the imaginary zeros of the L functions of the kind considered, lie on the critical line. This has been confirmed by the computation of the first zeros up to height $t = 10{,}000$ and $k \leq 13$; and up to $t = 2{,}500$ and k of a variety of forms, in ref. [49].

1.9. THE CASE OF THE HEILBRONN-DAVENPORT FUNCTION

The construction of this function f(s) is described in ref. [56] § 10.25, page 240-243, and can be defined as

$$f(s) = \tfrac{1}{2}\sec\theta\left\{e^{-i\theta}L_1(s) + e^{i\theta}L_2(s)\right\}$$

where $L_1(s)$ and $L_2(s)$ are Dirichlet L functions mod.5 and θ is such that

$$\tan 2\theta = 2\cos\left(\frac{2\pi}{5}\right)$$

and

$$\tan\theta = \frac{\sqrt{10-2\sqrt{5}} - 2}{\sqrt{5}-1} = 0,284...$$

(according to ref.[52], Titchmarsh puts erroneously above sin 2θ in change of tan 2θ)

f(s) has the functional equation

$$\left(\frac{5}{\pi}\right)^{s/2}\Gamma(1/2+s/2)\ f(s) = \left(\frac{5}{\pi}\right)^{1/2-s/2}\Gamma(1-s/2)\ f(1-s)$$

Putting again s = 1/2 + it, and equating arguments of both sides we deduce, as above, that:

$$\frac{t}{2}\log\left(\frac{5}{\pi}\right) + \arg\Gamma(3/4+it/2) + \arg f(1/2+it) =$$

$$= -t/2\ \log\left(\frac{5}{\pi}\right) + \arg\Gamma(3/4-it/2) + \arg f(1/2-it)\ \pm\ n\pi$$

or, equivalently

$$\pm n\pi = t\log 5/\pi + 2\arg\Gamma(3/4+t/2) + 2\arg f(1/2+it)$$

(1.9.1) $$\pm n\pi = -\ t/2\left(\log\frac{|5t|}{2\pi} - 1\right) - \pi/8 + O(1/t) + O(\log t)$$

(here we have used Backlund’s theorem of ref. [56] § 9.4, p. 180 that if

$$f(s) = O(t^A),$$

then $$\arg f(1/2+it) = O(\log t).$$

In ref. [52] have been computed the first 22 zeros of f(s) on the critical line. We cannot compute these zeros through formula (1.9.1) because we do not know the values of arg f (1/2 + it).

But we can compute in change, the values of the Gram points of f(s), defined by the formula

$$t \log 5/\pi \ +2 \arg \Gamma (3/4 + it/2) = \pm n\pi$$

and eventually one can check if between two Gram points g_m, g_{m+1} there is a zero. (Gram's "Law").

The results for $m \geq 7$, are exhibited in the following table:

γ_7 = 20,1597		γ_{15} = 37,4554	
	g_7 = 22,75		g_{15} = 38,2
γ_8 = 23,3453		γ_{16} = 40 ,1626	
	g_8 = 24,8		g_{16} = 40,35
γ_9 = 26,0949		γ_{17} = 40,682953	
	g_9 = 27,0		g_{17} = 41,6
γ_{10} = 27,9238		γ_{18} = 43,0813	
	g_{10} = 29,0		g_{18}= 44,0
γ_{11} = 30,1594		γ_{19} = 44,9471	
	g_{11} = 31,0		g_{19} = 45,7
γ_{12} = 31,4645		γ_{20} = 46,4563	
	g_{12} = 32,7		g_{20} = 47,2
γ_{13} =33,6998		γ_{21} =48,4778	
	g_{13} = 34,5		g_{21} = 49,25
γ_{14} = 35,8908		γ_{22} =50,2401	
	g_{14} =36,5		

It turns out to be that actually Gram's law is fulfilled, thus supporting indirectly the numerical validity of (1.9.1).

The numerical computations performed in ref. [52] put into light the existence of the following zeros off the critical line.

0,8085 + 85,6993 i	0,5743 + 166,4793 i
0,6508 + 114,1633 i	0,7242 + 176,7025 i

But at present nobody has computed N(T), the quantity of zeros in a rectangle in the critical strip, and so we are unable to determine how many zeros are outside the critical line.

In change, A. Karatsuba, in his most recent paper on f(s) (ref.[19]), has proved that No(T) > O (T (log T) $^{½ -\varepsilon}$)), in good agreement with (1.9.1) (N_0 (T) being the quantity of zeros on the critical line)

1.10. THE CASE OF EPSTEIN'S ZETA FUNCTIONS

An Epstein zeta function Z(s) is defined by the series development

(1.10.1) $$Z(s) = \sum_m \sum_n \frac{1}{(am^2 + bmn + cn^2)^s}$$

where a,b,c are real numbers, a>0, c > 0, $\Delta = 4ac - b^2 > 0$ and the summation is taken over all values of m and n except m=n=0.

The series converges as far as Re s > 1.

It has a functional equation which resembles that of the Riemann zeta function, namely:

(1.10.2) $$\xi^*(s) = \left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-s} \Gamma(s)\, Z(s) = \left(\frac{2\pi}{\sqrt{\Delta}}\right)^{s-1} \Gamma(1-s)\; Z(1-s) = \xi^*(1-s)$$

Here we put s = 1/2 + it and we obtain

(1.10.3) $$\Gamma (1/2 + it)\, Z (1/2 + it) = P^{2it}\, \Gamma (1/2 - it)\, Z (1/2 - it)$$

where $$P = \frac{2\pi}{\sqrt{\Delta}}$$

From (1.8.3) follows, equating arguments of both sides:

(1.10.4) $$\arg Z (½ + it) + \arg \Gamma (½ + it) - t \log P = \pm n\pi$$

valid for every t, which gives the position of the zeros along the critical line.

Replacing arg Γ (½ + it) by its asymptotic value we get:

(1.10.5) $$\arg Z (1/2 + it) = -t \log |t| + \left(1 + \log\frac{2\pi}{\sqrt{\Delta}}\right) t + O\left(\frac{1}{t}\right) \pm n\pi$$

Potter and Titchmarsh (ref [43]) have computed N(T), the quantity of zeros in the critical strip in the range $0 \le t \le T$, and found (by contour integration) that

$$N(T) = \frac{1}{\pi} T \log T - \frac{1}{\pi}\left(1 + \log\frac{2\pi\lambda}{\sqrt{\Delta}}\right) T + O(\log T) + O(1) + O\left(\frac{1}{T}\right)$$

Here

$$O(\log T) + O(1) \text{ stands for } \frac{1}{\pi}\arg Z\left(\frac{1}{2}+iT\right); \text{ and}$$

$$T\log T - T + O\left(\frac{1}{T}\right) \quad \text{stands for} \quad \arg\Gamma(1/2 + iT)$$

λ is the least value other than zero taken by $p(m, n) = am^2 + bmn + cn^2$.

Hence

$$\pi N(t) = \arg\Gamma(1/2 + it) + \arg Z(1/2 + it) - t\log P - t\log\lambda + O(1)$$

and there would be zeros on the critical strip every time that

(1.10.6) $$\pm n\pi = \arg\Gamma(1/2 + it) + \arg Z(1/2 + it) - t\log P - t\log\lambda + O(1)$$

It is clear that (1.10.4) and (1.10.6) differ only in a term $-t\log\lambda + O(1)$

Hence would result the following theorem:

Theorem: Z(s) has all its zeros on $\sigma = ½$, (with the exception of O(1) zeros), if and only if $\lambda = 1$. In any other case, there are $T\log\lambda + O(1)$ zeros outside the critical

line.

Besides, theorem 3 of ref. [43] states that if $\sqrt{4ac - b^2}$ is rational and congruent to 2 (mod 4) then $N_0(T) > K T^{1/2 + \varepsilon}$ in agreement with (1.10.5)

But very regrettably the results of ref. [19] are contradicted by the results in ref. [55], due to the Prof. Harald Stark .

We reproduce here textually Theorem 2 of his paper:

Let N(T,Q) denote the number of zeros of $\zeta(s,Q)$ in the region $-1 < \sigma < 2$, $0 \le t \le T$. If $k > K$ and $0 < T \le 2k$, then

$$N(T,Q) = \frac{T}{\pi}\log\left(\frac{kT}{\pi e}\right) + O\left\{\log^{1/2}(T+3)\left[\log\log(T+3)\right]^{1/6}\right\}$$

The constant implied by "O" is independent of k (and $k = \sqrt{\Delta}/2a$).

The contradiction arises from the fact that in ref. [43], N(T) depends on the quantity λ, while in ref.[55] it depends on the quantity k of the Q-polynomial.

This is why at present we can not arrive at any conclusion in the case of the Epstein series.

There are some cases where the Epstein series can be decomposed as product of other functions. For instance, we have that if

$$Q = u^2 + v^2, \qquad \text{then} \qquad Z_{Q(s)} = 4\,\zeta(s)\,L(s, \chi_{-4})$$

$Q = u^2 + 4v^2$ then $Z_{Q(s)} = 2\,\zeta(s)\,L(s, \chi_{-4})\,\{1-2^{-s}+2^{1-2s}\}$

$Q = u^2 + uv + 3v^2$ then $Z_{Q(s)} = 2\,\zeta(s)\,L(s, \chi_{-11})$

$Q = u^2 + 7v^2$ then $Z_{Q(s)} = 2\,\zeta(s)\,L(s, \chi_{-7})\,\{1-2^{1-s}+2^{1-2s}\}$

Where χ_k denotes the Kronecker symbol (k / *) (ref. [16])

Now, we remark that the zeros of

$$f(s) = 1-2^{1-s}+2^{1-2s}$$

are

$$s = \frac{1}{2} \pm \frac{\pi/4+2k\pi}{\log 2}\, i$$

and those of

$$g(s) = 1 - 2^{-s} + 2^{-2s}$$

are

$$s = \frac{1}{2} \pm \frac{2k\pi+\alpha}{\log 2}\, i \qquad \alpha = \text{arc tg}\sqrt{7}$$

Hence, all the imaginary zeros of the four above Z_Q (s) lie on the critical line, according to §4. Next, we remark that the four functions have in common that $\lambda = 1$.

1.11. RAMANUJAN'S FUNCTION AND SERIES

Ramanujan's function $\tau(n)$ is the coefficient of x^n in the series expansion

$$x\left\{(1-x)(1-x^2)(1-x^3)\ldots\right\}^{24} = \Sigma\, \tau(n)\, x^n$$

Ramanujan's series F(s) is the Dirichlet series

$$F(s) = \sum \frac{\tau(n)}{n^s}$$

which has the product decomposition

(1.11.1) $$F(s) = \prod_p \frac{1}{1-\tau(p)\,p^{-s}+p^{11-2s}}$$

F(s) has a very simple functional equation

(1.11.2) $$\xi(s) = (2\pi)^{-s}\,\Gamma(s)\,F(s) = (2\pi)^{s-12}\,\Gamma(12-s)\,F(12-s) = \xi(12-s)$$

From (1.11.1) and (1.11.2) follows that the critical strip of F(s) lies in the band $11/2 \le \sigma \le 13/2$, and the critical line is $\sigma = 6$

When, in the (1.11.2) we choose $s = 6 + it$, and we equate the arguments of both members, we obtain:

$$-t \log 2\pi + \arg \Gamma(6+it) + \arg F(6+it) = t \log 2\pi + \arg \Gamma(6-it) + \arg F(6-it) \pm 2n\pi$$

Hence, there are zeros along the critical line every time that

(1.11.3) $\pm n\pi = -t \log 2\pi + \arg \Gamma(6+it) + \arg F(6+it)$

This agrees with J.L.Hafner's result (1983) that $N_0(T) > AT \log T$

Concerning N(T), the quantity of zeros in the rectangle $11/2 \le \sigma \le 13/2$ $0 \le t \le T$, according to the ref. [20] we have:

$$N(T) = \{ + \arg \Gamma(6+iT) - T \log 2\pi + \arg f(6+iT) \}$$

It follows that there are zeros in the critical strip every time that

(1.11.4) $\pm n\pi = \arg \Gamma(6+it) + \arg f(6+it) - t \log 2\pi$

Then (1.11.4) coincides with (1.11.3) and we deduce that all the imaginary zeros lie on $\sigma = 6$.

The preceding conclusion can be confirmed using the fact that, analogously to the zeta function, we have that

$$f(6+it) = Z(t)\, e^{-i\vartheta(t)}$$

where

$$Z(t) = \Gamma(6+it)\, f(6+it)\,(2\pi)^{-it} \sqrt{\frac{\sinh(\pi t)}{\pi t(1+t^2)(4+t^2)(9+t^2)(16+t^2)(25+t^2)}} \quad \theta(t) =$$

$$= \arg \Gamma(6+it) - t \log(2\pi)$$

and Z(t) is real for real t.

Numerical computation has shown that the first 12069 imaginary zeros of F(s) lie on the critical line $\sigma = 6$ and are simple ones. Also have been calculated the first 5018 zeros and 2228 successive zeros beginning with the 20001 st. zero (ref. [20])

1.12. THE CASE OF THE FUNCTION $\zeta_n(s) = \sum_{x_1} \dots \sum_{x_n} \frac{1}{(x_1^2 + x_2^2 + \dots + x_n^2)^s}$

Here the x_i are positive integers, though they can be zero, but not all zero at the same time.

The series converges absolutely for $\sigma > n/2$. The function has a pole at $s = n/2$, with residue $\pi^{n/2} / \Gamma(n/2)$. The critical strip is defined by the interval $0< \sigma < n/2$, and the critical line is $\sigma = n/4$.

Its functional equation is

(1.12.1) $$\pi^{-s}\ \Gamma(s)\ \zeta_n(s) = \pi^{-(n/2-s)}\ \Gamma(n/2-s)\ \zeta_n\,(n/2-s)$$

We put here $s = n/4 + it$, and equate the arguments of both sides.

We obtain :

(1.12.2) $$\pm\ n\pi = -t \log\pi + \arg \Gamma\,(n/4+it) + \arg \zeta_n\,(n/4+it)$$

This formula gives the position of the zeros along the critical line.

If the methods of ref. [43] were correct, then there would be zeros on the critical line every time that

(1.12.3) $$- t \log \pi + \arg \Gamma\,(n/4+it) + \arg \zeta_n\,(n/4+it) – O(1) = \pm\, n\pi$$

(Of course, this is the same than (1.10.6) when we put there $P = \pi$ and $\lambda =1$)

Comparison of (1.12.2) and (1.12.3) would lead to the following.

Theorem 1.12.1. The function $\zeta_n(s)$ has all its zeros on $\sigma = n/4$, with O(1) possible exceptions off of it.

The existence of an O(1) term would open the possibility of one or several real zeros in the interval $0 < \sigma < n/2$

1.13. GENERALIZATION OF THE METHOD TO OTHER CASES

At this stage, it can be seen at what kind of Dirichlet series D(s) can be applied the preceding methods: primarily are those who have a functional equation of the type

$$D(s) = F(s)\ .\ D\,(1- s)$$

(or reducible to this form by a change of variable).

Putting there $s = 1/2 + it$, and equating arguments of both members, one can compute an expression that fixes the distribution of the zeros along the critical line.

Then one compares this with the result obtained by contour integration around the critical strip, and one sees if both distributions differ between them or coincide. This proves or disproves the “Riemann Hypothesis” for the specific function under consideration.

The former methods of proof have generated a strong feeling of refuse and opposition, based on the opinion that "It may not be possible that so complex and difficult problems can be solved in such a simple way."

This way of thinking has been swept by the actual facts, and it is pertinent to quote here the words of Gauss in 1849 in the introduction of Eisenstein's works: - Gauss wrote- "The higher arithmetic" (i.e.: Number Theory) "presents us with an inexhaustible store of interesting truths, too, which are not isolated, but stand in the closest relation to one another, and between which, with each successive advance of the science, we continually discover new and wholly unexpected points of contact. A great part of the theories of arithmetic derive on additional charm "*from the peculiarity that we easily arrive by induction at important propositions, which have the stamp of simplicity upon them, but the demostration of which lies so deep as not to be discovered until after many fruitless efforts; and even then it is obtained by some tedious and artificial process; while the simpler methods of proof long remain hidden from us*".

1.14. ANSWER TO SOME OBJECTIONS THAT HAVE BEEN RECEIVED

Objection (1.14.1): The fact that $N_0(T)$ be equal to N(T) does not prove the Riemann hypothesis. It could happen that for a given $T = \gamma_i$ could exist two zeros at this level : one on $\sigma = ½$ and other in $\sigma = 1/3$, for instance.

Answer: When in the expression for N(T), the variable T changes from

$$T = \gamma_i - 0 \text{ to } = \gamma_i + 0, \; N(T)$$

changes exactly in the same quantity than does $N_0(T)$. Hence, due to this very special form of N(T) and $N_0(T)$, such a possibility is entirely excluded.

Objection (1.14.2): In the proof of § 3,5 it is assumed that $\zeta(1/2 + it) = 0$ that is the thing that must be proved.

Answer: When one wishes to find the zeros of a polynomial P(x) at some stage of the reasoning one must put P(x) =0, or one is solving other problem.

It is obvious then that if one wishes to find the zeros of the zeta function on the critical line, one must put in some place that $\zeta(1/2 + it) = 0$,as was put in (1.5.3)

Objection (1.14.3): In equality (1.2.3) it is not proved that k be a bounded quantity.

Answer: It makes no sense to determine if k is bounded or not; in every case, equality (1.2.5) holds.

Objection (1.14.4): The same objection that 3.- , with the quantity r of (1.5.10)

Answer: In any case, (1.5.11) is valid.

Objection (1.14.5): Concerning the β of (1.5.5), it seems that lots of ordinates of zeros could give rise to the same β.

Answer: Who denies such a thing? And what importance has this fact with the calculation that is performed after (1.5.5)?

Objection (1.14.6): One γ_i could make in (1.5.4)

$$\sum_{n\leq m} \frac{1}{n^{n1/2-it}} + M\, e^{-i\alpha m} = 0 , \qquad (\rho_i = \beta_0 + \gamma_i)$$

in which case it is not counting at all, it seems.

Answer: When passing from (1.5.3) to (1.5.4) it was clearly stated that

$$t \neq t_i = 0$$

of the above expression. We can not assume a case that was discarded beforehand.

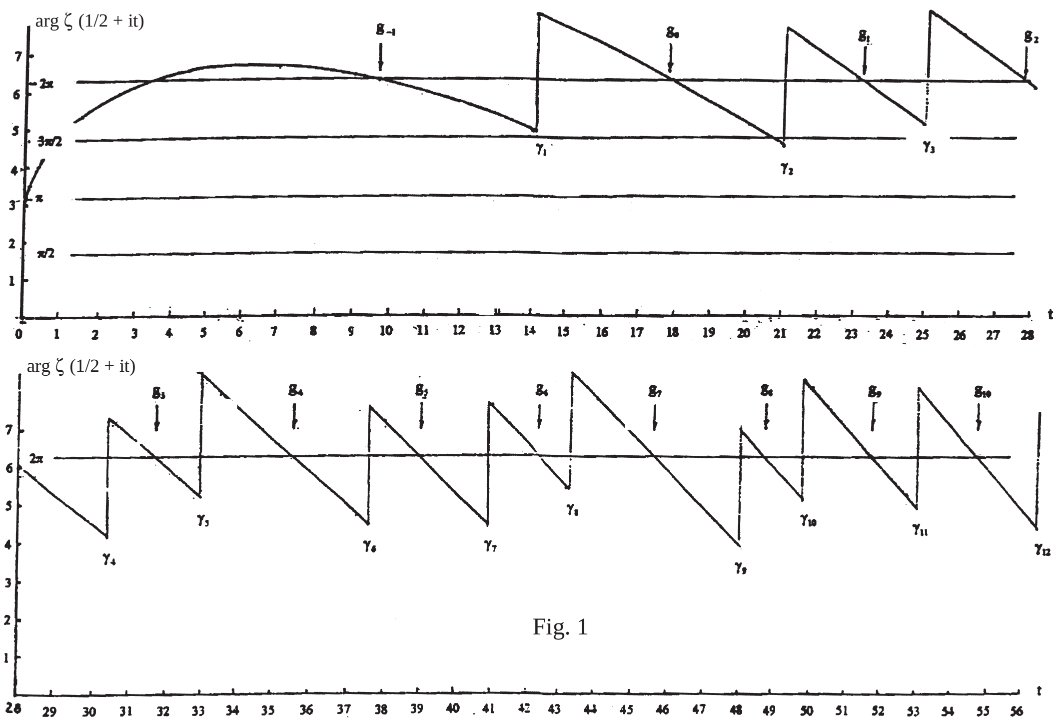

Fig. 1