CHAPTER 10

THE SINGULAR SERIES OF THE GENERALIZED FERMAT EQUATION

by Aldo Peretti

10.0. INTRODUCTION

In this chapter the results of Chapter 6 are used to give the infinitely many solutions for the generalized Fermat equation.

10.1. INTERSECTION FORMULA

We denote the generalized Fermat equation by the Diophantine equation

(10.1.1) $$x^a + y^b = z^c$$

This equation is of the form

$$y_3 = y_1 + y_2$$

and we wish to determine the number N(n) of solutions with

(10.1.2) $$y_3 = y_1 + y_2 \le n$$

Then, an intersection formula could be:

(10.1.3) $$N(n) = \sum \Delta([\sqrt[a]{y_1}]).\Delta([\sqrt[b]{y_2}]).\Delta([\sqrt[c]{y_3}])$$

where

(10.1.4) $$\Delta\left(\left[\sqrt[a]{y_1}\right]\right) = \left[\sqrt[a]{y_1}\right] - \left[\sqrt{y_1 - 1}\right] = \begin{cases} 1 \text{ if } u^a \le y_1 < u^a + 1 \\ 0 \text{ otherwise} \end{cases}$$

(with similar formulas for $[\sqrt[b]{y}]$ and $[\sqrt[c]{y}]$).

The sum in (10.1.2) extends to those values of y_1, y_2, y_3 such that (10.1.2) is fulfilled.

Actually , $\Delta([\sqrt[a]{y_1}])$ does not vanish only if y_1 is an a-th power, $\Delta([\sqrt[b]{y_2}])$ does not vanish only if y_2 is a b-th power, and $\Delta([\sqrt[c]{y_3}])$ does not vanish only if y_3 is a c-th power. In this case each term in (10.1.3) contributes with a unity to the sum.

The graph of N(n) consists of "squares" of width 1 and height 1 because of (10.1.4). Hence, if the limits of integration do not touch these "squares", we have:

(10.1.5) $$N(n)=\iint_{y_3=y_1+y_2\le n}\Delta\left(\left[\sqrt[a]{y_1}\right]\right).\Delta\left(\left[\sqrt[b]{y_2}\right]\right).\Delta\left(\left[\sqrt[c]{y_3}\right]\right)dy_1.dy_2.dy_3$$

In contrary case we have an asymptotic formula. Replacing y3 by its value y1+y2 we reduce it to a double integral:

(10.1.6) $$N(n)=\iint_{y_1+y_2\le n}\Delta\left(\left[\sqrt[a]{y_1}\right]\right).\Delta\left(\left[\sqrt[b]{y_2}\right]\right).\Delta\left(\left[\sqrt[c]{y_1+y_2}\right]\right)dy_1.dy_2$$

10.2. ASYMPTOTIC VALUE OF $\Delta\left(\left[\sqrt[a]{y1}\right]\right)$

As was shown in Chapter 9 of this book, we have:

(10.2.1) $$\Delta\left(\left[\sqrt[a]{y}\right]\right)\approx\frac{1}{a}\sum_{q=1}^{\infty}\frac{S(a,q,y)}{q}y^{\frac{1}{a}-1}$$

where

(10.2.2) $$S(a,q,y)=\sum_{\substack{h=0\\(h,q)=1}}^{q-1}w(a,q,h)e^{-2\pi iy\frac{h}{q}}$$

and

$$w(a,q,h)=\sum_{x=0}^{q-1}e^{2\pi 1x^a\frac{h}{q}}=\text{Weyl sum}$$

Of course, as $\Delta(\left[\sqrt[a]{y_1}\right])$ approximates a discontinuous function, the series (10.2.1) is conditionally convergent with very show rate of convergence.

(This is not the only formula we have available for $\Delta(\left[\sqrt[a]{y_1}\right])$ .Starting with the well known Fourier development

$$[x]=x-\frac{1}{2}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\sin(2\pi mx)}{m}$$

we could derive other similar formula)

10.3. ASYMPTOTIC FORMULA FOR N(n)

Next, we replace (10.2.2) in (10.1.6) and we obtain:

(10.3.1) $$N(n)=\iint_{y_1+y_2\le n} \frac{1}{a}\sum_{q=1}^{\infty}\frac{\sum_h w(a,q_1,h)}{q_1} e^{-2\pi i y_1 \frac{h_1}{q_1}} y_1^{\frac{1}{a}-1}$$

$$\frac{1}{b}\sum_{q_2=1}^{\infty}\frac{\sum_h w(b,q_2,h)}{q_2} e^{-2\pi i y_2 \frac{h_2}{q_2}} y_2^{\frac{1}{b}-1}.$$

$$\frac{1}{b}\sum_{q_3=1}^{\infty}\frac{\sum_h w(b,q_3,h)}{q_3} e^{-2\pi i (y_1+y_2) \frac{h_3}{q_3}} (y_1+y_2)^{\frac{1}{c}-1} dy_1.dy_2$$

But, as occurs in the binary Goldbach problem (chapter 6 of this book), the dominant term of the above expression is obtained when we choose $q_1=q_2=q_3=q$, $h_1=h_2=h_3=h$, and we have

(10.3.5) $$N(n)\approx\frac{1}{abc}\sum_q\frac{1}{q^3}\left\{\sum_h w(a,q,h)\sum_h w(b,q,h)\sum_h w(c,q,h)\right\}\cdot$$

$$\cdot \iint_{y_1+y_2\le n} e^{-4\pi i\frac{h}{q}(y_1+y_2)} (y_1+y_2)^{\frac{1}{a}-1}(y_1)^{\frac{1}{b}-1}(y_2)^{\frac{1}{c}-1} dy_1.dy_2$$

The integral I(n) in the second one is of the Dirichlet-Liouville type and can be evaluated by ref [27].

According to the theorem there we have:

(10.3.6) $$I(n)=\frac{\Gamma(1/a)\Gamma(1/b)}{\Gamma(1/a+1/b)}\int_0^n e^{-4\pi i t\frac{h}{q}} t^{1/a+1/b+1/c-2}dt$$

if $M = 1/a + 1/b + 1/c > 1$

Replacing this in (10.3.5) we obtain:

(10.3.7) $$N(n)\approx\frac{\Gamma(1+1/a)\Gamma(1+1/b)}{c\,\Gamma(1/a+1/b)}\int_0^n \sum_{q=1}^{\infty}\frac{\sum_h w(a,q,h)}{q}\cdot e^{-\frac{4}{3}\pi i t\frac{h}{q}}$$

$$\cdot\sum_{q=1}^{\infty}\frac{\sum_h w(b,q,h)}{q} e^{-\frac{4}{3}\pi i t\frac{h}{q}} . \sum_{q=1}^{\infty}\frac{\sum_h w(c,q,h)}{q} e^{-\frac{4}{3}\pi i t\frac{h}{q}} . t^{1/a+1/b+1/c-2}dt$$

This formula can be written also as:

(10.3.8) $$N(n) \approx \frac{\Gamma(1+1/a)\Gamma(1+1/b)}{c\,\Gamma(1/a+1/b)}\int_0^n S(a,q,t)\ S(b,q,t)\ S(c,q,t)t^{m-1}dt$$

where

$$S(a,q,t) = \sum_{q=1}^{\infty}\left\{w(a,q,h)e^{-\frac{4}{3}\pi it\frac{h}{q}}\right\}$$

with like formulas for S(b,q,t) and S (c,q,t), and m = 1/a+1/b+1/c

We apply now the first mean value theorem of the integral calculus:

$$\int_p^q f(x).g(x).d(x) = f(\xi)\int_p^q g(x).d(x) \qquad (p \le \xi \le q)$$

to (10.3.8) obtaining thus that

(10.3.9) $$N(n) \approx \frac{\Gamma(1+1/a)\Gamma(1+1/b)}{c\,\Gamma(1/a+1/b)} S(a,q,\xi)\ S(b,q,\xi)\ S(c,q,\xi)\frac{n^{M-1}}{M-1}$$

valid for M > 1.

The product f the three S's is evidently the singular series of the problem. (10.3.9) proves the existence of infinitely many solutions if m > 1. For m ≤ 1 nothing can be asserted.

Example (10.3.1):

The equation

$$x_1^2 + x_2^2 = z^c \le n$$

has

$$N(n) \approx \frac{\pi}{4} S^2(2,q,\xi)S(c,q,\xi)n^{1/c}$$

solutions.

10.4. GENERALIZATION TO THE DIOPHANTINE EQUATION $z^{a_0} = x_1^{a_1} + \ldots + x_m^{a_m}$

This is an equation of the type

(10.4.1) $$y_{m+1} = y_1 + y_2 + \ldots + y_m \le n$$

and we obtain, as a straightforward extension of (10.3.9) that:

(10.4.2) $$N(n) \approx \frac{\Gamma(1+1/a_1)..\Gamma(1+1/a_m)}{a_0\Gamma(1/a_1+...+1/a_m)} S(a_0,q,\xi)..S(a_m,q,\xi)\frac{n^{m-1}}{m-1}$$

with

(10.4.3) $m_1 = 1/a_1 + \ldots + 1/a_m$

valid $m_1 > 1$

Example (10.3.2):

For the Euler equation

(10.4.4) $$z^k = x_1^k + ... + x_m^k \le n$$

we have that

(10.4.5) $$N(n) \approx \frac{\Gamma^m(1+1/k)}{k\Gamma(m/k)} S^{m+1}(k,q,\xi)\frac{n^{\frac{m+1}{k}-1}}{\frac{m+1}{k}-1}$$

This formula is valid for (m + 1)/k > 1 or m > k – 1.

Hence the Euler equation always has some solution if m ≥ k : an unproved fact at present.

Remark that the Euler hypothesis that a k-th power is the sum of k k-th powers, today has been verified numerically only for k = 3, 4, 5, 7 and 8. All the attemps to incorporate k = 6 to the list have failed at present, spite the inmense power of modern computers.