and we obtain, as a straightforward extension of (10.3.9) that:

(10.4.2) $$N(n) \approx \frac{\Gamma(1+1/a_1)..\Gamma(1+1/a_m)}{a_0\Gamma(1/a_1+...+1/a_m)} S(a_0,q,\xi)..S(a_m,q,\xi)\frac{n^{m-1}}{m-1}$$

with

(10.4.3) $m_1 = 1/a_1 + \ldots + 1/a_m$

valid $m_1 > 1$

<u>Example (10.3.2):</u>

For the Euler equation

(10.4.4) $$z^k = x_1^k + ... + x_m^k \le n$$

we have that

(10.4.5) $$N(n) \approx \frac{\Gamma^m(1+1/k)}{k\Gamma(m/k)} S^{m+1}(k,q,\xi)\frac{n^{\frac{m+1}{k}-1}}{\frac{m+1}{k}-1}$$

This formula is valid for (m + 1)/k > 1 or m > k – 1.

Hence the Euler equation always has some solution if m ≥ k : an unproved fact at present.

Remark that the Euler hypothesis that a k-th power is the sum of k k-th powers, today has been verified numerically only for k = 3, 4, 5, 7 and 8. All the attemps to incorporate k = 6 to the list have failed at present, spite the inmense power of modern computers.

CHAPTER 11

PRIMES OF THE FORM $P^k = X_1^m + \ldots + X^m \le N$

by Aldo Peretti

11.0. INTRODUCTION

Using an appropiate intersection formula is deduced the asymptotic quantity of solutions of the Diophantine equation of the title.

11.1. INTERSECTION FORMULA

Denote with N(n) the quantity of solutions of the equation

(11.1.1) $$P = x_1^{\ m} + \ldots + x_r^{\ m}$$

where p denotes a prime number and the x_i are positive integer numbers.

Then we have the exact formula:

(11.1.2) $$N(n) = \sum \Delta\left(\left[\sqrt[m]{y_1}\right]\right) \ldots \Delta\left(\left[\sqrt[m]{y_r}\right]\right) \Delta\pi\left(y_{r+1}\right)$$

where $\Delta\left(\left[\sqrt[m]{y_1}\right]\right)$ has the meaning of (10.1.4) and $\Delta\pi(y)$ is defined as

$$\Delta\pi(y) = \pi(y) - \pi(y-1)$$

In fact each term in (11.1.2) does not vanish only if y_1 is a m–th power, … , y_r is also a m-th power, and

(11.1.3) $$y_{r+1} = y_1 + y_2 + \ldots + y_r$$

is a primer number

11.2 ASYMPTOTIC FORMULA

An asymptotic formula for $\Delta\pi(y)$ can be derived from the fact that

(11.2.1) $$\Delta\pi(y)=\frac{\Delta\vartheta(y)}{\log y}$$

(for every value of y) and the formula stated at the of the former Lemma (6.16.3). We deduced thus that

(11.2.2) $$\Delta\pi(y)\sim\sum_{q=1}^{N}\frac{\mu(q)}{\varphi(q)}\frac{c_q(t)}{\log\ t}$$

As regards $\Delta\left(\left[\sqrt[m]{y}\right]\right)$, according to the former formula (9.3.12) we have:

(11.2.3) $$\Delta\left(\left[\sqrt[m]{y}\right]\right)\sim\frac{t^{1/m-1}}{m}\sum_{q=1}^{N}\sum_{h=0}^{q-1}\frac{w(m,q,h)}{q}e^{-2\pi iy\, h/q}$$

Replacing in (11.1.2) the values of the Δ´s given by (11.2.2) and (11.2.3), and the multiple sum by an integral, we get

$$N(n)\sim\int\limits_{y_1+\ldots+y_r=y_{r+1}\le n}\!\!\!\ldots\int\Delta\left(\left[\sqrt[m]{y_1}\right]\right)\ldots\Delta\left(\left[\sqrt[m]{y_r}\right]\right).\Delta\pi\left(y_{r+1}\right)dy_1\ldots dy_{r+1}$$

$$\sim\int\ \ldots\int\frac{1}{m^r}\sum_{q_1}\sum\frac{w(m,q_1,h_1)}{q_1}e^{-2\pi i\frac{h_1}{q_1}y_1}.y_1^{1/m-1}\times\ldots$$

(11.2.4) $$\times\sum_{q_r}\sum_{h}\frac{w(m,q_r,h_r)}{q_r}e^{-2\pi i\frac{h_r}{q_r}y_r}\ y_r^{1/m-1}\times$$

$$\times\sum_{q_{r+1}}\frac{\mu\left(q_{r+1}\right)}{\varphi\left(q_{r+1}\right)}\frac{c_q\left(y_{r+1}\right)}{\log y_{r+1}}\times dy_1\ldots dy_{r+1}$$

The dominant term in the integrand is obtained when $q_1 = q_2 = \ldots = q_{r+1}$, so that

$$N(n) \sim \int_{y_1+...+y_r=y_{r+1}\le n} ... \int \frac{1}{m^r} \sum \sum \frac{w^r(m,q,h)}{q^r} \frac{\mu(q) c_q(y_{r+1})}{\varphi(q) \log y_{r+1}} \times$$

(11.2.5)

$$\times e^{-2\pi i \frac{h}{q}(y_1+...+y_r)} . y_1^{1/m - 1} ... y_r^{1/m - 1} dy_1 ... dy_{r+1}$$

We replace here y_{r+1} by $y_1 + y_2 + ... + y_r$, and we get:

$$N(n) \sim \int_{y_1+...+y_r\le n} ... \int \frac{1}{m^r} \sum \sum \frac{w^r(m,q,h)}{q^r} \frac{\mu(q) c_q(y_1+...+y_r)}{\varphi(q) \log(y_1+...+y_r)} \times$$

(11.2.6)

$$\times e^{-2\pi i \frac{h}{q}(y_1+...+y_r)} . y_1^{1/m - 1} ... y_r^{1/m - 1} dy_1 ... dy_r$$

By the Dirichlet – Lioville theorem this is equal to:

(11.2.7) $$N(n) \sim \frac{\Gamma^r(1+1/m)}{\Gamma(r/m)} \int_r^n \sum_q \sum_h \frac{w^r(m,q,h)}{q^r} \frac{\mu(q)}{\varphi(q)} e^{-2\pi i \frac{h}{q} t} \frac{c_q(t)}{\log(t)} t^{r/m - 1} dt$$

and takinng account of the first mean value theorem of the integral calculus:

(11.2.8) $$N(n) \sim \frac{\Gamma^r(1+1/m)}{\Gamma(r/m)} \sum_q \sum_h \left(\frac{w^r(m,q,h)}{q^r} \right) e^{-2\pi i \frac{h}{q} \xi} \frac{\mu(q)}{\varphi(q)} c_q(\xi) \int_r^n \frac{t^{r/m - 1}}{\log t} dt$$

Hence, the equation always has solutions if $r \ge m$, if the singular series does not vanish.

11.3. COMPARISON WITH KNOWN RESULTS, AND CONSEQUENCES.

- Conjecture N of Hardy-Littlewood states that in the case of

$$p = x_1^3 + x_2^3 + x_3^3$$

holds that

$$N(n) \sim \Gamma^3(4/3) \frac{n}{\log n} \prod_p \left(1 - \frac{A_p}{p^2}\right)$$

where A_p depends on the cubic residues of p.

The relation of $\sum_{q}\sum_{h}\frac{w^{3}(3,q,h)}{q^{3}}e^{-2\pi i\frac{h}{q}t}$ with the cubic residues can be consulted in ref [5].

- According to Lagrange every prime number is the sum of four (or fewer) squares.
- About half of the primes (those of the form 4n + 1), are the sum of two squares.
- An inmediate consequence of (11.2.8) is that the quantity of primes that are sums of r r-th powers is a fraction of the totality of primes.