CHAPTER 12

TRIPLETS AND OTHER SEQUENCES OF PRIMES

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12.0. INTRODUCTION

The numbers x, x + 2, x + 4 can not be simultaneously primes, because some of the three values must be multiple of three.

In change, the triplets x, x + 2, x + 6 or x, x + 4, x + 6 can be assume actually prime values simultaneously.

Hence, the problem now arises of how many primes N(n) are in the m-plet sequence

(12.0.1) $$x + a_1 \; ; \; x + a_2 \; ; \; \ldots \; ; \; x + a_m \leq n$$

where the a_i take only "admissible" values.

Hardy and Littlewood dedicated many pages to the solution of this problem in ref. [11].

12.1. A FORMULA FOR N(n)

We have that asymptotically

(12.1.1) $$N(n) \approx \int_2^n \Delta\pi(x + a_1).\Delta\pi(x + a_2)..\Delta\pi(x + a_m).dx$$

where

$$\Delta\pi(y) = \pi(y) - \pi(y-1) = \begin{cases} 1 \text{ if } p < y < p+1 \\ 0 \quad \text{otherwise} \end{cases}$$

because the $\Delta(x + a_i)$ in the integrand do not vanish only if every number $x + a_i$ is prime.

But in Chapter 6 of this book it has been proved that

$$\Delta\vartheta(y) \approx \sum_{q=1}^{\infty} \frac{\mu(q)}{\varphi(q)} Cq\,(y)$$

so that

$$\Delta\pi(y) \approx \sum \frac{\mu(q)}{\varphi(q)} \frac{Cq\ (y)}{\log\ y}$$

Replacing in (12.1.1) we get

(12.1.2)
$$N(n) \approx \int_2^n \left\{ \sum_{q=1}^{n} \frac{\mu(q)}{\varphi(q)} \frac{Cq\,(x + a_1)}{\log(\,x + a_1)} \right\} \ldots \left\{ \sum_{q=1}^{n} \frac{\mu(q)}{\varphi(q)} \frac{Cq\,(x + a_m)}{\log(\,x + a_m)} \right\} dx$$

If our a_i are small compared with x, then we can put

(12.1.3)
$$N(n) \approx \int_2^n \left\{ \sum \frac{\mu(q)}{\varphi(q)} Cq_1(x + a_1) \right\} \ldots \left\{ \sum \frac{\mu(q)}{\varphi(q)} Cq_m(x + a_m) \right\} \frac{dx}{\log^m x}$$

In a similar way to what happens in binary Goldbach problem of Chapter 6, Lemma (6.5.2), the dominant term in the integrand is obtained when we take $q_1 = q_2 = .. = q_m$, so that

(12.1.4)
$$N(n) \approx \int_2^n \sum \frac{\mu^m(q)}{\varphi^m(q)} Cq(x + a_1) \ldots Cq(x + a_m) \frac{dx}{\log^m x}$$

Now the first mean value theorem of the integral Calculus states that

$$\int_p^q f(x).g(x).dx = f(\xi) \int_p^q g(x).dx$$

if f(x) is a continuous function, g(x) is an integrable function; $p \le \xi < q$; $m \le f(x) \le M$ and g(x) does not change sign in the interval (p, q).

Adopting

$$g(x) = \frac{1}{\log^m x} \qquad\qquad f(x) = Cq(x + a_1) \ldots Cq(x + a_m)$$

we deduce:

(12.1.5)
$$N(n) \approx \sum_{q=1}^{\infty} \frac{\mu^m(q)}{\varphi^m(q)} Cq(\xi + a_1) \ldots Cq(\xi + a_m) \int_2^n \frac{dx}{\log^m x}$$

The singular series can be easily transformed into a product along the primes and we obtain:

(12.1.6)
$$N(n) \approx \prod_p \left\{ \frac{\mu^m(q)}{\varphi^m(q)} Cq(\xi + a_1) \ldots Cq(\xi + a_m) \right\} \int_2^n \frac{dx}{\log^m x}$$

The ξ numbers are comprised between 2 and n. If we suppose that some of them are integer, then we have, as known, that

$$Cp(\xi + a_i) = \begin{cases} -1 & \text{if } p \mid \xi + a_i \\ p - 1 & \text{if } p \mid \xi + a_i \end{cases}$$

Then (account taken that μ(p) = -1 and φ(p) = p – 1) (12.1.6) turns out to be:

$$N(n) \approx \prod_{p \mid \xi + a_m} \left\{ 1 + \frac{(-1)^{m+1}}{(p-1)^m} \right\} \prod_{p \chi \xi + a_m} \left\{ 1 + \frac{(-1)^{m+1}}{(p-1)^m} \right\} \int_2^n \frac{dx}{\log^m x}$$

The Hardy-Littlewood conjecture was that

$$N(n) \approx \prod_{p>2} \left(\frac{p}{p-1} \right)^{m-1} \frac{p-v}{p-1} \int_0^n \frac{dx}{\log^m x}$$

where v is the number of distinct residues of $a_1, a_2, \ldots, a_m$ to modulus p.