CHAPTER 2

AN ALTERNATIVE PROOF USING THE FUNCTION $N(\sigma,T)$

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2.0. INTRODUCTION

This is another proof of the Riemann hypothesis, along entirely different approach, in order to be more convinced about the first proof.

Starting with a formula of Littlewood, use is made of another formula due to von Mangoldt in order to evaluate [2.1.1] at the beginning of the paper. It is concluded that it vanishes for every T. which implies the truth of the Riemann hypothesis.

2.1. LITTLEWOOD'S FORMULA

Denote with N(σ,T) the quantity of imaginary zeros $\rho_n = \beta_n + i\gamma_n$, of the zeta function with $\beta_n \geq \sigma$, and $T \geq t$, $s = \sigma + it$, as usual. Then we have:

(2.1.1) $$\int_{\sigma_o}^{1} N(\sigma,T)d\sigma = \sum_{\beta n>\sigma_o} (\beta_n - 1/2) \qquad (\sigma_0 \geq 1/2)$$

with $0 \leq \gamma_n < T$. (ref. [56])

According to a formula due to Littlewood, (ref. [56]) holds that:

(2.1.2) $$2\pi \int_{\sigma_O}^{1} N(\sigma,T)\,do = \int_{o}^{T} \log|\zeta(\sigma_o + it)|dt - \int_{o}^{T} \log|\zeta(2+it)|dt + \int_{\sigma_O}^{T} \arg\zeta(\sigma_o + it)\,do + k(\sigma_o)$$

We have, besides, the following expansion, due to von Mangoldt, and reproduced in ref. [23] (formula IV):

(2.1.3) $$\mathrm{Log}\,\zeta(s) = \sum_{n\leq x} \frac{\Lambda_1(n)}{n^2} + \left\{\mathrm{Li}\left(x^{1-s}\right) \pm \pi i\right\} + \sum\left\{\mathrm{Li}\left(x^{\rho-s}\right) \pm \pi i\right\} + \sum_{q=1}^{\infty}\left\{\mathrm{Li}\left(x^{-2q-s}\right) \pm \pi i\right\}$$

Here Li (x) denotes the logarithmic-integral function; the dash in the first sum at right means that if x is a natural number, the last term in the sum must be halved; Λ1 (n) = Λ (n)/log n, where Λ (n) is von Mangoldt's function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^{\alpha} \\ 0 & \text{otherwise} \end{cases}$$

$x \geq 1$ and the formula is valid for every s.

Besides, the + sign, must be taken if the corresponding Li has negative ordinate in the exponent, and the – sign contrary case. Remark that all the brackets are real quantities for real s

Putting

(2.1.4) $$Li_1(x^{\alpha}) = Li(x^{\alpha}) \pm \pi i$$

we can write the preceding formula as:

(2.1.5) $$\log \zeta(s) = \sum_{n \leq x} \frac{\Lambda_1(n)}{n^s} - Li_1\left(x^{1-s}\right) + \sum_r Li_1\left(x^{r-s}\right)$$

where r denotes either an imaginary or real zero of the zeta function.

As a consequence of (2.1.5) we deduce:

(2.1.6) $$\arg \zeta(\sigma + it) = \operatorname{Im} \log \zeta(s) = -\sum_{n \leq x} \frac{\Lambda_1(n)}{n^{\sigma}} \sin(t \log n) - \operatorname{Im} Li_1\left(x^{1-s}\right) + \operatorname{Im} \sum_r Li_1\left(x^{r-s}\right)$$

Here Im denotes the imaginary part. Analogously:

(2.1.7) $$\log\left|\zeta(\sigma + it)\right| = \sum \frac{\Lambda_1(n)}{n^{\sigma}} \cos(t \log n) - \operatorname{Re} Li_1\left(x^{1-s}\right) + \operatorname{Re} \sum_r Li_1\left(x^{r-s}\right)$$

The limits in ∑' have been omitted for the sake of simplicity, and Re denotes the real part.

We recall also that

(2.1.8) $$Li_1(x^{\alpha}) = \int_0^x \frac{u^{\alpha-1}}{\log u} du$$

if Re $\alpha \geq o$ and the path is a straight line between o and x, with an indentation above or under u = 1, as the case may be.

When Re $\alpha < o$, we have that

(2.1.9) $$Li_1(x^{\alpha}) = \int_{\infty}^x \frac{u^{\alpha-1}}{\log u} du$$

We shall use, besides, the function $Li_2(x^{\alpha})$ defined by

(2.1.10)
$$\mathrm{Li}_1(x^{\alpha}) = \begin{cases} \int_0^x \frac{u^{\alpha-1}}{\log^2 u} & \text{if } \mathrm{Re}(\alpha) > 0 \\ \int_\infty^x \frac{u^{\alpha-1}}{\log^2 u} du & \text{if } \mathrm{Re}(\alpha) < 0 \end{cases}$$

with similar paths of integration than (2.1.8).

2.2. TWO AUXILIARY FORMULAS

We shall need still two formulas that are an immediate consequence of the above. Firstly, we have:

$$\mathrm{Im}\int_{\sigma_o}^{2} \mathrm{Li}_1(x^{\alpha-s})d\sigma = \mathrm{Im}\int_{\sigma_o}^{2} d\sigma\int_0^x \frac{u^{\alpha-1-s}}{\log u}du = \mathrm{Im}\int_0^x \frac{u^{\alpha-1-it}}{\log u}du\int_{\sigma_o}^{2} u^{-\sigma}d\sigma =$$
$$= \mathrm{Im}\int_0^x \frac{u^{\alpha-1-it}}{\log^2 u}\left\{\frac{1}{u^{\sigma_0}} - \frac{1}{u^2}\right\} = \mathrm{Im}\,\mathrm{Li}_2(x^{\alpha-\sigma_0-it}) - \mathrm{Im}\,\mathrm{Li}_2(x^{\alpha-2-it})$$

This is to say:

(2.2.1)
$$\mathrm{Im}\int_{\sigma_o}^{2} \mathrm{Li}_1(x^{\alpha-s})d\sigma = \mathrm{Im}\,\mathrm{Li}_2(x^{\alpha-\sigma 0-it}) - \mathrm{Im}\,\mathrm{Li}_2(x^{\alpha-2-it})$$

This has been proved with the use of formula (2.1.8); but we arrive equally well to the same result if we use (2.1.9).

Besides, in analogous way, we deduce:

(2.2.2)
$$\mathrm{Re}\int_0^T \mathrm{Li}_1(x^{r-\sigma_0-it})\,dt = \mathrm{Im}\,\mathrm{Li}_2(x^{r-\sigma_0-it}) + \mathrm{Im}\,\mathrm{Li}_2(x^{r-\sigma_0})$$

2.3. EVALUATION OF T_1

Let us calculate now:

$$\int_{\sigma_0}^{2} \arg\zeta(\sigma+it)d\sigma$$

The first series at the right hand side of (2.1.6) can be integrated with respect to σ because it is a finite sum. The second series when r = ρ, is uniformly convergent with respect to x between discontinuities at x = p^m, and uniformly convergent with respect to s for all values of s (assuming t ≠ o).

When r = -2q = real zeros, the series is evidently uniformly convergent for all values of σ.

Hence we can integrate them term by term over a finite interval in which the integrals have a meaning, and we deduce:

(2.3.1) $$\int_{\sigma_0}^{2} \arg\ \zeta(\sigma+it)d_0 = -\sum \Lambda_1(n)\sin\ (T\log n)\int_{\sigma_0}^{2}\frac{d\sigma}{n^{\sigma}} - \operatorname{Im}\int_{\sigma_0}^{2}\operatorname{Li}_1\left(x^{1-s}\right)d\sigma +$$

$$+\operatorname{Im}\sum_r \int_{\sigma_0}^{2} \operatorname{Li}_1(x^{r-s})d\sigma =$$

$$= -\sum{}'\Lambda_1(n)\frac{\sin(T\log n)}{\log n}\left\{\frac{1}{n^{\sigma_0}}-\frac{1}{n^2}\right\} - \operatorname{Im}\operatorname{Li}_2(x^{-1-it}) +$$

$$+\operatorname{Im}\sum_r \operatorname{Li}_2(x^{r-\sigma_0-it}) + \operatorname{Im}\sum_r \operatorname{Li}_2\left(x^{r-2-it}\right)$$

where use has been made of (2.2.1)

2.4. EVALUATION OF T_2

We evaluate now:

$$\int_0^T \log\left|\zeta(\sigma_0+it)\right|dt$$

We integrate here formula (2.1.7) with respect to t, on the same grounds as before, and we get:

(2.4.1) $$\int_0^T \log\left|\sigma_0+it\right|dt = \sum\frac{\Lambda_1(n)}{n^{\sigma_0}}\frac{\sin(T\ \log\ n)}{\log n} - \operatorname{Re}\int_0^T \operatorname{Li}_1(x^{1-\sigma_0-it})dt +$$

$$+\operatorname{Re}\sum_r \int_0^T \operatorname{Li}_1(x^{r-\sigma_0-it})dt =$$

$$=\sum{}'\frac{\Lambda_1(n)}{n^{\sigma_0}}\frac{\sin(T\ \log\ n)}{\log n} + \operatorname{Im}\operatorname{Li}_2(x^{1-\sigma_0-it}) - \operatorname{Im}\operatorname{Li}_2(x^{1-\sigma_0}) -$$

$$-\operatorname{Im}\sum_r \operatorname{Li}_2(x^{1-\sigma_0-iT}) + \operatorname{Im}\sum_r \operatorname{Li}_2(x^{1-\sigma_0})$$

where use has been made of (2.2.2).

Next, we put $\sigma_o = 2$ and subtract from (2.4.1) the resulting expression.

We get:

(2.4.2)
$$\int_0^T \log|\zeta(\sigma_0+it)|dt - \int_0^T \log|\zeta(2+it)|dt =$$

$$= \sum \Lambda_1(n)\frac{\sin(T\log n)}{\log n} + \left\{\frac{1}{n^{\sigma_0}} - \frac{1}{n^2}\right\} - \text{Im}\,\text{Li}_2(x^{1-\sigma_0-iT}) + \text{Im}\,\text{Li}_2(x^{-1-iT}) -$$

$$-\text{Im}\,\text{Li}_2(x^{1-\sigma_0}) + \text{Im}\,\text{Li}_2(x^{-1}) - \sum_r \text{Im}\,\text{Li}_2(x^{r-\sigma_0-iT}) +$$

$$+\text{Im}\sum_r \text{Li}_2(x^{r-2-iT}) + \text{Im}\sum_r \text{Li}_2(x^{r-\sigma_0}) - \text{Im}\,\text{Li}_2(x^{r-2})$$

According to (2.3.1), this is equal to:

(2.4.3)
$$\int_{\sigma_0}^2 \arg\zeta(\sigma+iT)do - \text{Im}\,\text{Li}_2(x^{1-\sigma_0}) + \text{Im}\,\text{Li}_2(x^{-1}) -$$

$$+\text{Im}\sum_r \text{Li}_2(x^{r-\sigma_0}) + \text{Im}\sum_r \text{Li}_2(x^{r-2})$$

Hence:

(2.4.4)
$$\int_o^T \log|\zeta(\sigma_o+it)|dt - \int_o^T \log|\zeta(2+it)|dt + \int_{\sigma_o}^2 \arg\zeta(\sigma+iT)do =$$

$$= -\text{Im}\,\text{Li}_2(x^{1-\sigma_o}) + \text{Im}\,\text{Li}_2(x^{-1}) + \text{Im}\sum_r \text{Li}_2(x^{r-\sigma_o}) - \text{Im}\sum_r \text{Li}_2(x^{r-2})$$

The right hand side depends on σ_o, x, and the r’ s, that are constants, but does not depend on T. Hence, its value is 0 (1).

2.5. THE FINAL FORMULA

We return now to (2.1.2), deducing that

(2.5.1)
$$\int_{\sigma_0}^1 N(\sigma,T)d\sigma = 0(1) = \text{absolute constant}$$

The value of the constant can be determined by giving to T an arbitrary value. For instance, it is known that the zeta function has not any zero if t < 14.

Hence, the 0 (1) constant is zero, and this implies the truth of the Riemann hypothesis.

The best evaluation of (2.5.1) known at present seems to be that of Selberg:

$$\int_{\sigma_0}^1 N(\sigma,T) = 0(T)$$