CHAPTER 3

SOLUTION OF GOLDBACH'S CONJECTURE

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3.0. INTRODUCTION

An exact expression is derived for the Hardy- Littlewood function v(t) by means of the Laplace transform. The application of the generalized final value theorem allows us to prove a result that essentially is the hypothetical asymptotic formula for the Hardy-Littlewood function v(t), with corresponding remainder term.

Goldbach's conjecture is an open problem from 1742, when Christian Goldbach, in a letter to Euler enunciated it as a conjecture that "every even number greater than two can be expressed as the sum of two primes".

However Descartes had conjectured it before Goldbach, so that at present it is in fact misnamed.

3.1. DERIVATION OF A BASIC FORMULA

Using the Chebishev function $\vartheta(u)$ defined by

(3.1.1) $$\vartheta(u) = \sum_{p \le \{u\}} \log p$$

where p denotes the prime numbers, we have:

(3.1.2) $$\Delta\vartheta(u) = \vartheta(u) - \vartheta(u-1) = \begin{cases} \log p & \text{if } [u] = p \\ 0 & \text{in any other case} \end{cases}$$

Hence, if we form

(3.1.3) $$v(t) = \sum \Delta\vartheta(u_1) \sum \Delta\vartheta(u_2)$$

where the sums are extended to those values of u_1 and u_2 such that

$$t = u_1 + u_2$$

evidently, $\nu(t)$ is the Hardy – Littlewood function

(3.1.4) $$\nu(t) = \sum_{p_1 + p_2 = t} \log p_1 \log p_2$$

Immediately, from (3.1.3) we can see that we also have

(3.1.5) $$\nu(t) = \sum_{u_1<t} \Delta\vartheta(u_1) \sum_{u_2<t-u_1} \Delta\vartheta(u_2)$$

or

(3.1.6) $$\nu(t) = \sum_{u_1 \le t} \Delta\vartheta(u_1) \sum_{u_1 \le t} \Delta\vartheta(t-u_1)$$

In fact, in the first sum of (3.1.5) we choose arbitrarily any integer $u_1 < t$; while in the second we are forced to restrict the sum to the $u_2 < t-u_1$.

As ϑ (u) is a step function the sums on the right-hand side of (3.1.6) are also step functions, that can be replaced by the corresponding integrals, so that we can write:

(3.1.7) $$\nu(t) = \int_0^x \Delta\vartheta\ (u+1)\ du \int_0^x \Delta\vartheta\ (t-u)\ du$$

The passage from formula (3.1.6) to (3.1.7) is given in full detail in Chapter 4, Section 2 of this book.

Evidently, the right hand member is, by definition, the convolution of the two functions.

(3.1.8) $$\nu(t) = \Delta\vartheta(u) * \Delta\vartheta\ (u+1)$$

with current notation.

Consequently, if we take the Laplace transform L of both sides, by usual rules, we obtain

$$L\{\nu(t)\} = L\{\Delta\vartheta(u+1)\}\ L\{\Delta\ \vartheta(u)\} =$$

$$= L^2\ \{\Delta\vartheta(u)\}.e^{-s}$$

(3.1.9) $$L\ \{\nu(t)\} = L^2\ \{\Delta\vartheta(u)\}.e^{-s}$$

and this is our basic formula needed.

3.2. THE EVALUATION OF L $\{\Delta\vartheta(u)\}$

From the definition of the Laplace transform we deduce:

(3.2.1) $$L\ \{\ \vartheta\ [\ u\ ]\ \} = \frac{1}{s} \sum \log p.e^{-ps}$$

From this it follows that

(3.2.2) $$L\left\{\Delta\vartheta(u)\right\} = \frac{1-e^{-s}}{s}\sum_{p}\log p\ e^{-ps}$$

when we take into account (3.1.2) | x | - 1

3.3. THE POLES OF f (s) = $\sum_{p}\log\ pe^{-ps}$

Setting $e^{-s} = x$ we have :

(3.3.1) $$\sum_{p}\log p\ e^{-ps} = \sum_{p}\log p\, x^{p} = F(x)$$

It is known that F (x) has infinitely many simple poles on the circle $|x| = 1$.

Hence, f (s) has infinitely many poles on the line σ = Re (s) = 0

For further calculation using with the Laplace transforms, we are interested in the residues of these poles at $x = e^{2\pi ih/q}$ as a primitive root of unity of order q. Consequently, we must assume that (h,q) =1

The answer is given by theorem 248, Ch 5 , p.212 of [25]

(3.3.2) $$F(x) \sim \frac{\mu(q)}{\varphi(q)\ (1 - x\, e^{-2\pi ih/q})} = -\frac{\mu(q)\, e^{2\pi ih/q}}{\varphi(q)\,(x - e^{2\pi ih/q})}$$

where μ (q) (is Möbius function) = $(-1)^{r}$ if q has r factors and $\varphi(q)$ (Euler`s function) is the number of integers $\leq$ q and coprime with q.

More exactly, as h can take the values h = 0, 1, 2, ... q –1 with the restriction (h, q) = 1, we can write that F(x), in the surroundings of the φ (q) primitive roots of unity of order q, is as

(3.3.3) $$F(x) \sim -\sum_{h=0,\,(h,q)=1}^{q-1}\frac{\mu(q)\ e^{2\pi ih/q}}{\varphi(q)\ (x - e^{2\pi ih/q})}$$

When we extend this reasoning to all the roots of order q $\leq$ N, we get the approximation

(3.3.4) $$F(x) \sim -\sum_{q=1}^{N}\ \sum_{h=0,\,(h,q)=1}^{q-1}\frac{\mu(q)\ e^{2\pi ih/q}}{\varphi(q)\,(x - e^{2\pi ih/q})}$$

If we desire an error term for this approximation, we can use the remainder term given by theorem 248 of [25]. However, for our purposes it is sufficient to put in evidence the number and the kind of the poles of F(x) only.

Next if we put $x = e^{-s}$ in (3.3.4), we obtain:

(3.3.5) $$F(e^{-5}) = f(s) \sim -\sum_{q=1}^{N} \sum_{h=0,(h,q)=1}^{q-1} \frac{\mu(q)\, e^{2\pi ih/q}}{\varphi(q)\,(e^{-s} - e^{2\pi ih/q})}$$

Again here we are interested in the calculation of the poles of f (s), and in the value of their residues.

We have that the residue of

$$g(s) = \frac{1}{e^{-s} - e^{2\pi ih/q}}$$

at the simple pole $s = -2\pi i h / q$ is

$$- e^{-2\pi i h / q}$$

Thus we can write (3.3.5) alternatively as

(3.3.6) $$f(s) \sim \sum_{q=1}^{N} \sum_{h=0,(h,q)=1}^{q-1} \frac{\mu(q)}{\varphi(q)\,(s+2\pi ih/q)}$$

The right hand side of the formula (3.3.6) is known as the Farey dissection of order N of f(s).

Replacing (3.3.6) in (3.2.2) we get:

(3.3.7) $$L\{\Delta\vartheta(u)\} \sim \frac{1-e^{-s}}{s} \sum_{q=1}^{N} \sum_{h=0,(h,q)=1}^{q-1} \frac{\mu(q)}{\varphi(q)\,(s+2\pi ih/q)}$$

and taking into account formula (3.1.9) and (3.3.7) we have:

(3.3.8) $$L\{\nu(t)\} \sim e^{-s} . \left(\frac{1-e^{-s}}{s}\right)^2 \left\{ \sum_{q=1}^{N} \sum_{h=0,(h,q)=1}^{q-1} \frac{\mu(q)}{\varphi(q)\,(s+2\pi ih/q)} \right\}^2$$

If we wish to pass from the asymptotic sign to the equality sign, we must write:

(3.3.9) $$L\{\nu(t)\} = e^{-s} . \left(\frac{1-e^{-s}}{s}\right)^2 \left\{ \lim_{N\to\infty} B_N^2(s) \right\}$$

where $B_N(s)$ is given in the second bracket on the right hand side of (3.3.8). The importance of formula (3.3.8) lies in the fact that it exhibits the poles of $L\{\nu(t)\}$, which is the basis for the evaluation of $\nu(t)$ by the residues method.

3.4. ABELIAN AND TAUBERIAN THEOREMS CONCERNING THE LAPLACE TRANSFORM

Suppose that $f(s) = L\{F(u)\}$ and $g(s) = L\{G(u)\}$ and →

(3.4.1) $f(s) \sim g(s)$ when $s \to 0^+$

means that

$$\sum_{q>N}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t) = \sum_{d/t} \frac{\mu^2(d)}{j(d)} \cdot \sum_{N/d<q,(q,t)=1} \frac{\mu(q)}{j^2(q)}$$

and similarly

(3.4.2) $F(t) \sim G(t)$

when $t \to \infty$ means that

$$\lim_{t\to\infty} F(t)/G(t) = 1$$

An implication from (3.4.2) to (3.4.1) is valid under rather general assumptions (that are fulfilled in our case), and is called an “Abelian theorem”.

In the theory of the Laplace transform it is called the generalized theorem of the final value, which also is valid even for discontinuous functions.

The converse implication from (3.4.1) to (3.4.2) is true under additional assumptions only, and it is named a “Tauberian Theorem”, which is available a wide variety of cases.

However, we shall not appeal to such kind of theorems.

3.5. APPLICATION OF THE ABELIAN THEOREM TO (3.3.8)

Let

$$\nu^*(t) = \lim_{t\to\infty} \nu(t)$$

such that

$$\nu^*(t) \sim \nu(t)$$

Then, by the Abelian theorem we get:

$$L\left\{\nu^*(t)\right\} = \lim_{t\to\infty} L\left\{\nu(t)\right\} = \lim_{s\to 0} e^{-s}.\left(\frac{1-e^{-s}}{s}\right)^2 B_N^2(s)$$

where $B_N(s)$ is as defined before in (3.3.8)

This reduces to :

(3.5.1) $$L\left\{\nu^*(t)\right\} = B_N^2(s)$$

from which it follows

(3.5.2) $$\nu^*(t) = L^{-1}\left\{B_N^2(s)\right\}$$

More exactly, the passage from (3.5.1) to (3.5.2) should by written as

$$(3.5.3) \qquad \nu_m^*(t) = \frac{\nu^*(t+0) + \nu^*(t-0)}{2} = L^{-1}\left\{B_N^2(s)\right\}$$

because ν∗(t) is a discontinuous function

3.6. EVALUATION OF $\nu_m^*(t)$ ACCORDING TO (3.5.3)

Due to a well known theorem [51], we have :

$$(3.6.1) \qquad \nu_m^*(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} B_N^2(s)\, e^{ts}\, ds$$

where the contour between $c \pm i\infty$ is a Bromwich contour, that contains all the poles of $B_N{}^2(s)$.

In the usual way, it is evaluated as the sum of the residues of the integrand at their poles, within the Bromwich contour.

Now, we have that :

$$(3.6.2) \qquad B_N^2(s) = \sum_{q=1}\sum_{h=0,(h,q)=1} \frac{\mu^2(q)}{\varphi^2(q)\,(s+2\pi ih/q)^2} +$$

$$+\sum_{q_1=1}^{N}\sum_{h_1=0,(h_1q_1)=1}^{q_1-1}\sum_{q_2=1}^{N}\sum_{h_2=0,(h_2,q_2)=1}^{q_2-1} \frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)\,(s+2\pi ih_1/q_1)\,(s+2\pi ih_2/q_2)} =$$

$$= \sum_{q=1}^{N}\sum_{h=0,(h,q)=1}^{q-1} \frac{\mu^2(q)}{\varphi^2(q)\,(s+A)^2} +$$

$$+\sum\sum\sum\sum \frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)}\,\frac{1}{(A_2-A_1)}\left\{\frac{1}{s+A_1}-\frac{1}{s+A_2}\right\}$$

with the obvious notation for A, A_1, A_2 and the sums.

The residues of the integrand at the double poles $1/(s+A)^2$ are easily found to be

$$\frac{\mu^2(q)}{\varphi^2(q)}\; t\; e^{-At}$$

The residue of the integrand at the simple pole $1/(s+A_1)$ is found to be

$$\frac{\mu(q_1)}{\varphi(q_1)}\; e^{-A_1 t}$$

Hence, collecting partial results, we have :

(3.6.3) $$v_m^*(t) = = \sum\sum \frac{\mu^2(q)}{\varphi^2(q)} \; t \; . \; e^{-At} +$$

$$+ \sum\sum\sum\sum \frac{\mu(q_1)\;\mu(q_2)}{\varphi(q_1)\;\varphi(q_2)} \cdot \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1}$$

In a more simplified way, we can argue alternatively that the inverse Laplace transform of the finite sums in (3.6.2) are the sums of the inverse Laplace transform of its terms, and after that we appeal to the tables.

The quadruple sum in (3.6.3) can be maximized as follows: The minimal value of $A_2 - A_1$ is $2\pi i\,(h_1/q_1 - h_2/q_2)$ where both fractions are contiguous Farey fractions.

It is known [47] that

$$\min \frac{h_1}{q_1} - \frac{h_2}{q_2} = \frac{1}{N(N-1)}$$

so that

$$\left|\frac{1}{A_2 - A_1}\right| \leq \frac{N(N-1)}{2\pi}$$

Furthermore

$$\left| e^{-A_1 t} - e^{-A_2 t} \right| \leq 2$$

Hence:

$$\left| \sum_{h_1=0,(h_1,q_1)=1}^{q_1-1} \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| < \sum_{\substack{h_1=0,(h,q)=1\\(h_1,q_1)=1}}^{q_1-1} \left| \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| \leq \frac{\varphi(q_1)\,N(N-1)}{\pi}$$

with a similar result for the sum along h_2.

It follows from the preceding inequalities:

$$\left| \sum_{q_1}\sum_{h_1}\sum_{q_2}\sum_{h_2} \frac{\mu(q_1)\;\mu(q_2)}{\varphi(q_1)\;\varphi(q_2)} \; \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| < < \sum_{q_1=1}^{N}\sum_{q_2=1}^{N} \frac{1}{\pi^2} N^2 (N-1)^2 =$$

$$= \frac{1}{\pi^2} N^4 (N-1)^2 < \frac{N^6}{\pi^2}$$

So that (3.6.3) can be written as :

(3.6.4) $$v_m^*(t) = \sum_{q=1}^{N} \sum_{h=0,(h,q)=1}^{q-1} \frac{\mu^2(q)}{\varphi^2(q)} \; e^{-At}.t + \delta \frac{N^6}{\pi^2}$$

where $-1 < \delta < 1$. But

$$\sum_{h=0,(h,q)=1}^{q-1} e^{-At}$$

is Ramanujan’s sum Cq (t).

Hence (3.6.4) can be written as:

(3.6.5) $$\nu_m^*(t) = \sum_{q=1}^{N} \frac{\mu^2(q)}{\varphi^2(q)} Cq(t).t + \delta_1 \frac{N^6}{\pi^2}$$

where $-1 < \delta_1 < 1$.

3.7. THE EVALUATION OF $\nu_m^*(t)$ AND $f_N(t)$

Once we have evaluated

$$\nu_m^*(t) = L^2 \{ B_N(s) \}$$

it will be easy to evaluate

(3.7.1) $$f_N(t) = L^{-1} \left\{ \left(\frac{1-e^{-s}}{s} \right) B_N(s) \right\}^2$$

whose connection with (3.3.9) is evident

Let

$$L^{-1}(f_1(s)) = F_1(t) \quad \text{and} \quad L^{-1}(f_2(s)) = F_2(t)$$

where

(3.7.2) $$f_1(s) = \frac{(1-e^{-s})^2}{s^2} \quad \text{and} \quad f_2(s) = B_N^2(s)$$

Then $F_2(t) = \nu_m^*(t)$, as was seen before

In regard to

$$f_1(s) = \left(\frac{1-e^{-s}}{s} \right)^2$$

and from the tables of [46], Vol. 5 p. 59 formula (15) we obtain:

(3.7.3) $$F_1(t) = L^{-1} \left\{ \frac{(1-e^{-s})}{s^2} \right\} = \begin{cases} t & \text{if } 0<t<1 \\ 2-t & \text{if } 1<t<2 \\ 0 & \text{otherwise} \end{cases}$$

By the convolution theorem for the inverse Laplace transform we have:

$$f_N(t) = L^{-1}\{ f_1(s).f_2(s) \} = \int_0^t F_1(u)\ F_2(t-u)\,du$$

Replacing the values given by (3.7.2) and (3.7.3) in the above transform we obtain:

(3.7.4) $$f_N(t) = \int_0^t F_1(u)\,\nu_m^*(t-u)\,du =$$

$$= \int_0^1 u\,\nu_m^*(t-u)\,du + \int_1^2 (2-u)\ \nu_m^*(t-u)\,du$$

3.8. FINDING THE LIMIT OF $\mathbf{v}_m^*(t)$ WHEN N$\to\infty$

Now we find:

(3.8.1) $$\eta(t) = \lim_{N\to\infty} \nu_m^*(t)$$

For this we need:

Lemma (3.8.1). This lemma proves:

(3.L1) $$\left| \sum_{q=N+1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} Cq(t) \right| < 10240\ t^{\varepsilon}\ \frac{\log^3 N}{N}$$

Proof of lemma (3.8.1) According to [57], p.35 formula (3.23) we have:

(3.L2) $$\sum_{q>N}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} Cq(t) = \sum_{d/t} \frac{\mu^2(d)}{j(d)} \cdot \sum_{N/d<q,\,(q,t)=1} \frac{\mu(q)}{j^2(q)}$$

Now

(3.L3) $$\sum_{N/d<q,\,(q,t)=1} \left| \frac{\mu(q)}{\varphi^2(q)} \right| < \sum_{N/d<q,\,(q,t)=1} \frac{1}{\varphi^2(q)} < \sum_{N/d<q} \frac{1}{\varphi^2(q)}$$

But

(3.L4) $$\varphi(n) > \frac{n}{2+\log n} > \frac{n}{8\log n}$$

for every n > 1 [21], so that

(3.L5) $$\sum_{N/d<q} \frac{1}{\varphi^2(q)} < 64 \sum_{N/d<q} \frac{\log^2 q}{q^2} < 64 \int_{N/d-1}^{\infty} \frac{\log^2 u}{u^2}\,du$$

and

$$\int_Q^{\infty} \frac{\log^2 u}{u^2}\,du = \frac{\log^2 Q}{Q} + \frac{2\log Q}{Q} < 3\,\frac{\log^2 Q}{Q} \qquad \text{if} \quad Q > e^2$$

Hence

$$\int_{N/d-1}^{\infty} \frac{\log^2 u}{u^2}\, du = \int_{N/d-1}^{N/d} \frac{\log^2 u}{N/d}\, du + \int_{N/d}^{\infty} \frac{\log^2 u}{u^2}\, du <$$

(3.L6) $$\frac{\log^2 (N/d - \vartheta)}{(N/d - \vartheta)^2} + 3\,\frac{\log^2 (N/d)}{N/d} < 5\,\frac{\log^2 (N/d)}{N/d}$$

where $0 < \vartheta < 1$ and the last inequality holds for every N/d

Replacing it in (3.L5) yields:

(3.L7) $$\sum_{N/d<q} \frac{1}{\varphi^2 (q)} < 320\, \frac{\log^2 (N/d)}{N/d}$$

From (3.L2), (3.L3), and (3.L7) follows:

$$\left| \sum_{N<q}^{\infty} \frac{\mu^2 (d)}{\varphi^2 (q)} Cq(t) \right| < \sum_{d/t} \frac{\mu^2 (d)}{\varphi (d)}\; 320\, \frac{\log^2 (N/d)}{N/d} <$$

(3.L8) $$< 320 \sum_{d/t} \frac{8 \log d}{d} \frac{\log^2 (N/d)}{N/d} = 2560 \sum_{d/t} \frac{\log d\, \log^2 (N/d)}{N} <$$

$$< 2560 \sum_{d/t} \frac{\log^3 (N/d)}{N} < 2560\, d(t)\, \frac{\log^3 N}{N}$$

where d (t) denotes the quantity of divisors of t.

According to [25], holds that

(3.L9) $$d(t) < t^{\varepsilon}$$

Replacing this value in (3L8), we obtain (3L1).

Combining (3.6.5) and Lemma (3.8.1), we get

(3.8.2) $$\nu_m^* (t) = \sum_{q=1}^{\infty} \frac{\mu^2 (q)}{\varphi^2 (q)} Cq(t).\, t + \delta_1 \frac{N^6}{\pi^2} + \delta_2\, 10240 \frac{\log^3 N}{N} t^{1+\varepsilon}$$

If we choose

$$N = 5{,}0246\; t^{1/7 + \varepsilon}$$

then the second and third terms are equal (except for the δ´s)

In order to round the calculations we adopt

(3.8.3) $$N = 5\; t^{1/7 + \varepsilon}$$

Then (3.8.1) becomes

(3.8.4) $$\eta (t) = \sum_{q=1}^{\infty} \frac{\mu^2 (q)}{\varphi^2 (q)} Cq(t).t + \delta_1 \frac{5^6}{\pi^2} t^{6/7} +$$

$$+\delta_2 \frac{10240}{5} t^{6/7} \left(\log 5 + \varepsilon \log t \right)^3$$

Thus, in our evaluation we have included every possible pole at the primitive roots of unity.

Now

$$\delta_1 \frac{5^6}{\pi^2} t^{6/7} + \delta_2 \frac{10240}{5} t^{2/7} \left(\log 5 + \varepsilon \log t \right)^3 < \delta_4\, 3600\ t^{6/7} \log^3 t$$

(-1 < δ_i <1) for $t \geq 6$

Hence (3.8.4) reduces to;

(3.8.5) $$\eta(t) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} Cq(t).t + \delta_4\, 3600\ t^{6/7} \log^3 t$$

We know that due to the multiplicative character of its terms, the infinite series can be transformed into an infinite product, and therefore by [14] the final result is:

(3.8.6) $$\sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} Cq(t) = 2 \prod_{p=3}^{\infty} \left(1 - \frac{1}{(p-1)^2} \right) \prod_{p/t} \frac{p-1}{p-2} =$$

$$= 1{,}3203 \prod_{p/t} \frac{p-1}{p-2}$$

Thus $\eta(t)$ consists of a first dominant discontinuous term

(3.8.7) $$D(t) = 1{,}3203 \prod_{p/t} \frac{p-1}{p-2}\, t$$

and a second remainder term C (t), for which we have the bound

(3.8.8) $$C(t) = \delta_4\, 3600\ t^{6/7} \log^3 t$$

with

(3.8.9) $$\eta(t) = D(t) + C(t)$$

Recall that on account of (3.5.3) and (3.8.1) we have:

$$\eta(t) = \lim_{N \to \infty} L^{-1} \left\{ B_N^2(s) \right\}$$

3.9. BOUNDS FOR $\nu(t)$

From (3.7.4), letting $N \to \infty$ due to (3.7.1) and (3.3.9) we obtain:

$$\lim_{N \to \infty} f_N(t) = \frac{\nu(t+0) + \nu(t-0)}{2} = \int_0^1 u\, \eta(t-u)\, du +$$

$$+\int_1^2 (2-u)\ \eta(t-u)\,du$$

Here we apply (3.8.9) in order to obtain :

(3.9.1)
$$\frac{\nu(t+0)+\nu(t-0)}{2} = D(t)\int_0^1 u.du + D(t-1)\int_1^2 (2-u)\,du\ + +$$

$$+\int_0^1 u\,C(t-u)\,du + \int_1^2 (2-u)\ C(t-u)\,du$$

The last line can be evaluated by the mean value theorem of integral calculus, and we have

$$\frac{\nu(t+0)+\nu(t-0)}{2} = \frac{D(t)+D(t-1)}{2} + \delta_5\,C(t-\delta_5) + (1+\delta_6)C(t-1-\delta_6) =$$

$$= \frac{D(t+0)+D(t-0)}{2} + 3\ \delta_7\,C(t)\ ,$$

$0<\delta_7<1,\ 0<\delta_5,\ \delta_6<1$

Hence:

$$\nu(t) = D(t) + \frac{3}{2}\ \delta_7\,C(t)$$

Appealing again to (3.8.7) and (3.8.8) for t even, we have:

$$\nu(t) = 1{,}303\,t\ \prod_{p/t}\frac{p-1}{p-2}\,t\ +1{,}5\ \ \delta_7\ \ \delta_4\,3600\ \ t^{6/7}\log^3 t$$

from which follows the lower bound

(3.9.2) $\nu(t) > 1{,}303\ t\ -\ 5400\ \ t^{6/7}\log^3 t$

for every $t \geq 6$.

At this stage we can ask ourselves which is the greatest value that $\nu(t)$ can assume in the case that there is only one solution.

This solution can range from $p_1 = t-3$ $p_2 = 3$ up to $p_1 = t/2$ $p_2 = t/2$. Hence

$$\max \nu(t) = \log^2\ t/2,$$

or, better, since we must take into account the order (because the solution $p_1 + p_2$ is counted differently from $p_2 + p_1$) . Thus $\max\nu(t) = 2\log^2 t/2$

Consequently, if we know that

(3.9.3) $\nu(t) > 2\log^2 t/2$

we can be sure that there is at least one solution.

Replacing the lower bound (3.9.2) in (3.9.3) we have:

(3.9.4) $$1,303\ t - 5400\ t^{6/7}\ \log^3 t > 2\ \log^2 t/2$$

which must hold if there is at least one solution.

Now, it is a numerical fact that (3.9.4) holds for $t > 10^{75}$.

The existence of the solutions has been proved up to $t = 4 \,.\, 10^{11}$ [50].

3.10. CONCLUSION

Using the same technics, we can prove that every sufficiently large odd number > 5 is the sum of three primes.

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