CHAPTER 4

CLARIFICATIONS OF THE AUTHOR'S PREVIOUS RESULT ON GOLDBACH'S CONJECTURE

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4.0. INTRODUCTION

In Chapter 3 the author give a tentative proof a Goldbach's conjecture. The purpose of this result is to provide more explanations of the author's work in Chapter 3.

4.1. THE RELATION BETWEEN ϑ (u) AND v (t)

Starting with Chebishev's function ϑ (u):

(4.1.1)
$$\vartheta(u) = \sum_{p \le [u]} \log p$$

where p denotes the prime number, we have:

(4.1.2)
$$\Delta\vartheta(u) = \vartheta(u) - \vartheta(u-1) \begin{cases} \log p \text{ if } [u] = p \\ o \text{ in any other case} \end{cases}$$

Hence, if we form

(4.1.3)
$$v(t) = \sum_{t = u_1 + u_2} \Delta\vartheta(u_1)\, \Delta\vartheta(u_2)$$

Evidently v(t) is the Hardy – Littlewood function

(4.1.4)
$$v(t) = \sum_{p_1 + p_2 = t} \log p_1 \,.\, \log p_2$$

From (4.1.3) we have:

(4.1.5)
$$v(t) = \sum_{u_1 < t} \Delta\vartheta(u_1)\, \Delta\vartheta(t - u_1)$$

4.2. TRANSFORMATION OF THE SUM IN (4.1.5) INTO AN INTEGRAL

The $\Delta\,\vartheta$ (u) function, when is not zero, can be drawn as segments of length 1 at heights log p every time that u = p. More exactly, we have:

(4.2.1) $\Delta\vartheta(u) = \log p$

Whenever $p \le u < p + 1$ (otherwise it is zero). This is shown as shaded bars in fig. 4.2.1:

$\vartheta(u) - \vartheta(u - 1)$

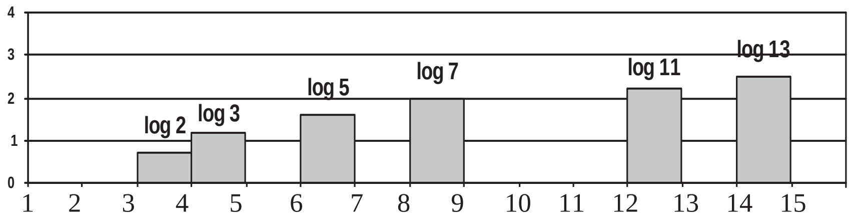

Fig. 4.2.1.

The graph of ϑ (t-u) is obtained from that of ϑ (u) by rotating the graph about the axis u = t / 2. In fig. 4.2.2 and 4.2.3 are shown the cases of

$$y = \vartheta(12 - u) - \vartheta(12 - u - 1) \text{ and } y = \vartheta(10 - u) - \vartheta(10 - u - 1).$$

$\vartheta(u + 1) - \vartheta(u)$

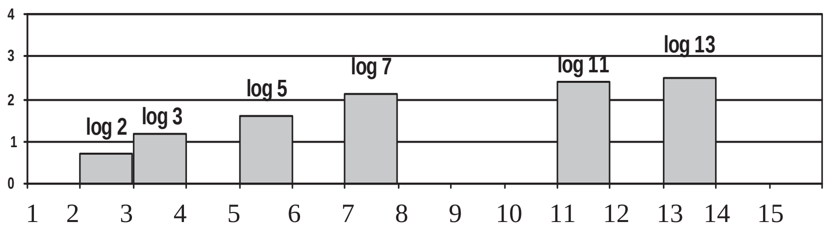

Fig. 4.2.2

$\vartheta(12 - u) - \vartheta(12 - u - 1)$

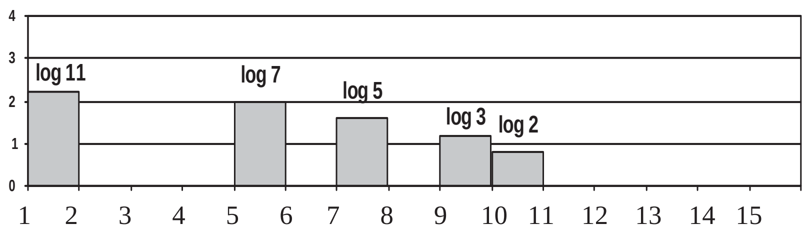

Fig. 4.2.3

Hence the graph of ϑ (u) - ϑ (u - 1) consists of unitary bars at the right of u = p, and that of ϑ (t - u) - ϑ (t - u - 1) consists of unitary bars at the left of u = p. If we form now the product

(4.2.2) $$\{\vartheta (u) - \vartheta (u -1)\} \{\vartheta (t -u) - \vartheta (t -u-1)\}$$

It is zero everywhere, except at the points where u = p_1 t – u = p_2, where it has the value

$$\log p_1 . \log p_2$$

It is evident from the graph that if we wish to obtain this value as an area, we must displace the graph of $\Delta \vartheta$ (u) to the left in one unity, because then:

(4.2.3)

$$\int_{p-1}^{p} \{\vartheta(u+1)-\vartheta(u)\}\{\vartheta(t-u)-\vartheta(t-u-1)\} = \log p . \log(t-p)$$

(The shaded areas of log p_1 and log p_2 multiply between themselves). This is shown in fig. 4.2.4 and 4.2.5. The integrand in (4.2.3) vanishes if u < 0 or u > t

ϑ (10 – u) – ϑ (10 – u – 1)

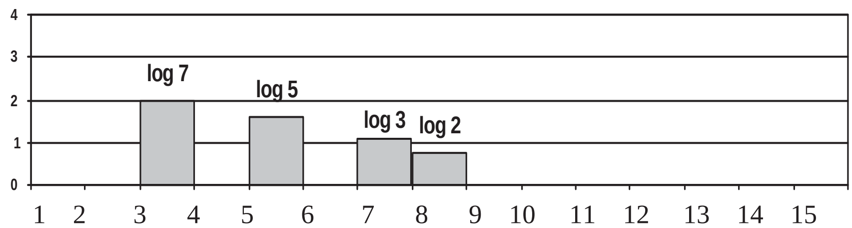

Fig. 4.2.4

{ϑ (u + 1) – ϑ (u)} {ϑ (12 – u) – ϑ (12 – u – 1)}

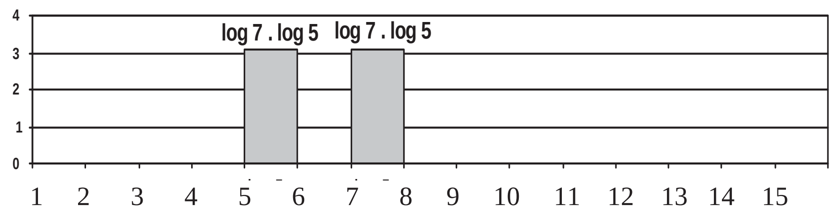

Fig. 4.2.5

Hence, we have that

(4.2.4) $$v(t) = \int_{o}^{t} \Delta\vartheta\,(u\ +1)\,\Delta\vartheta\,(t-u)\,du$$

This is nothing but the convolution of the two functions that appear at right.

Consequently, if we take the Laplace transform of both sides we obtain, by known rules:

(4.2.5)

$$L\{v(t)\} = L\{\Delta\vartheta(u+1)\}\,.\,L\{\Delta\vartheta(u)\}$$

By the second shift property of the Laplace transform we have:

$$L\ \{\Delta\vartheta\ (u\ +1)\} = e^{s}\ L\{\Delta\vartheta\ (u)\ \}$$

So that (4.2.5) transforms to

(4.2.6) $$L\{v(t)\} = e^{s} L^{2}\{\Delta\vartheta(u)\}$$

Or also:

(4.2.7) $$L\{v(t-1)\} = L^{2}\ \{\Delta\vartheta\,(u)\}$$

This will be used in § 11 later.

4.3. THE EVALUATION OF L { Δ ϑ (u)}

From the definition of the Laplace transform we deduce:

(4.3.1) $$L\ \{\ \vartheta\ (u)\ \} = \frac{1}{s} \sum_{p} \ \log p\ .\ e^{-ps}$$

From here follows that

(4.3.2) $$L\{\Delta\vartheta\ (u)\} = \frac{1-e^{-s}}{s} \ \sum_{p} \log p\,.\,e^{-ps}$$

(p: prime numbers) on account of (4.1.2)

By ref. [25], theorem 248 p. 212, we have:

(4.3.3) $$F(x) = \sum\ \log p\,.\,x^{p} = \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q} \frac{\mu(q)\,e^{2\Pi i h/q}}{\varphi(q)(x - e^{2\Pi i h/q})} + An^{\vartheta\,+1/4\,+\varepsilon}$$

(h, q) = 1.

As a consequence of this we have that

(4.3.4)
$$F(s) = \sum \log p . e^{-ps} = \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q} \frac{\mu(q)}{\varphi(q)(s + 2\pi\ i\, h/q)} +$$
$$+ An^{\vartheta + 1/4 + \varepsilon}$$

(see ref. [35] for a detailed calculation).

This formula is unconditional, and ϑ denotes the upper bound of the real part of the imaginary zeros of the L – series involved. (Here Re (s) = σ = 1/n: but this restriction does not affect our further reasoning).

Let us analyze formula (4.3.4). The function F (s) has a double infinitude of poles on the line σ = 0, whenever s = -2 πi h/q where 1≤ h ≤ q and 1≤ q ≤ ∞. They form a natural boundary, and the series at right of (4.4) describes the influence of the poles with q ≤ [√n], while the other term at right accounts for the influence of the poles with q > [√n].

It is evident that the better is the knowledge we have about the zeros of the L-series, the smaller is the remainder term in (4.3.4). According with Chapter 3 of this book, Section 1.8 we can adopt ϑ = ½ .

4.4. THE USE OF (4.3.4)

Replacement of (4.3.4) in (4.3.2), and of (4.3.2) in (4.2.6) yields:

(4.4.1)
$$L\{v(t)\} = e^{s}\left(\frac{1-e^{-s}}{s}\right)^{2}\left\{\sum_{q=1}^{[\sqrt{n}]}\sum_{h=1}^{q}\frac{\mu(q)}{\varphi(q)(s+2\pi\ i h/q)} + An^{\vartheta+1/4+\varepsilon}\right\}^{2}$$

or

(4.4.2)
$$v_m(t-1) = \frac{v(t-1+0)+v(t-1-0)}{2} = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^{2}\sum_{q}^{[\sqrt{n}]}\sum_{h=1}^{q}\frac{\mu^{2}(q)}{\varphi^{2}(q)(s+2\pi i h/q)^{2}} + \right.$$
$$+\left(\frac{1-e^{-s}}{s}\right)^{2}\sum\sum_{q_1\neq q_2}\sum\sum\frac{\mu(q_1)\mu(q_2)}{\varphi(q_1)\varphi(q_2)(s+2\pi i h_1/q_1)(s+2\pi i h_2/q_2)} +$$
$$\left.+\left(\frac{1-e^{-s}}{s}\right)^{2}\sum_{q=1}^{[\sqrt{n}]}\sum_{h}\frac{\mu(q)\left(2An^{\vartheta+1/4+\varepsilon}\right)}{\varphi(q)(s+2\pi i h/q)} + \left(\frac{1-e^{-s}}{s}\right)^{2} An^{2\vartheta+2/4+2\varepsilon}\right\} =$$

(4.4.3) $$= L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 \{T_1(s) + T_2(s) + T_3(s)\}\right\} + T_3$$

For reasons that will become evident in Section 4.9., we shall evaluate in first place $L^{-1}\{T_1(s)\}$ and $L^{-1}\{T_2(s)\}$.

According to the evaluations performed in ref. [25] we can adopt for A the value

(4.4.4) $$A = 80$$

(This value however is not relevant for what follows)

4.5. THE EVALUATION OF $L^{-1}\{T_1(s)\}$

Appealing to tables we have:

$$L^{-1}\{T_1(s)\} = L^{-1}\left\{\sum_{q=1}^{N}\sum_{h}\frac{\mu^2(q)}{\varphi^2(q)\,(s+2\pi i\, h/q)^2}\right\} =$$

(4.5.1) $$= \sum_{q=1}^{N}\sum_{h=q}^{q-1}\frac{\mu^2(q)}{\varphi^2(q)}e^{-2\pi i\, ht/q}\, t = \sum_{q=1}^{N}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t).t$$

where

$$C_q(t) = \sum_{\substack{h=0\\(h.q)=1}}^{q-1} e^{-2\pi i\, h\, t/q}$$

is Ramanujan’s function, and N = [√n]

According to Lemma 1 we have

$$\left|\sum_{q>n}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t)\right| < d(t)\ e^{3\gamma}\ \frac{(\log\ \log\ n)^2}{N}\ \log\ \log\ t$$

Hence in (4.5.1) we can put

(4.5.2) $$L^{-1}\{T_1(s)\} = \sum_{q=1}^{\infty}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t).t +$$

$$+\delta_1 e^{3\gamma} d(t)\frac{(\log\log N)^2}{N}\log\log t.t$$

Now the singular series

(4.5.3)

$$S(t) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t)$$

vanishes for odd t, and for even t can be transformed into an infinite product.
(ref. [57]).

(4.5.4)

$$S(t) = 2\prod_{p=3}^{\infty}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p|t}\frac{p-1}{p-2}\,t = 1{,}3203 \prod_{p|t}\frac{p-1}{p-2}\,.\,t = P(t)\,.\,t$$

which puts in evidence that S(t) is a discontinuous function. We have computed thus the influence of the infinitude of double poles in (4.4.2).

4.6. THE EVALUATION OF $L^{-1}\{T_2(s)\}$

According to tables we have:

(4.6.1)

$$F_5(t) = L^{-1}\{T_2(s)\} = L^{-1}\left\{\sum_{q_1=1}^{N}\sum_{h_1}\sum_{q_2=2}\sum_{h_1}\frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\varphi(q_2)(s+2\pi i\ h_1/q_1)\pi}\cdot\right.$$

$$\left.\cdot\ \frac{1}{(s+2\pi i\ h_2/q_2)}\right\} =$$

$$= \sum_{q_1=1}^{N}\sum_{h_1}\sum_{q_2=2}^{N}\sum_{h_2}\frac{\mu(q_1)\,\mu(q_2)\,(e^{-A_1 t} - e^{-A_2 t})}{\varphi(q_1)\,\varphi(q_2)\,A_2 - A_1}$$

($A_2 = 2\,\pi i\ h_2/q_2$, $A_1 = 2\,\pi i\ h_1/q_1$)

Taking into account Lemma (4.14.1), we have:

(4.6.2)

$$\left| L^{-1}\{T_2(s)\}\right| = \frac{\delta_2}{4\pi} N^2 (N+1)^2$$

We have computed thus the influence of the simple poles with $q \le N$ multiplied by themselves.

4.7. THE EVALUATION OF $L^{-1}\{T_3(s)\}$

We have:

(4.7.1) $$T_3 = An^{2\vartheta + 1/2 + 2\varepsilon} \,.\, L^{-1}\left(\frac{1-e^{-s}}{s}\right)^2$$

According to tables (ref. [46]), we have:

(4.7.2) $$F(t) = L^{-1}\left(\frac{1-e^{-s}}{s}\right)^2 = \begin{cases} t & \text{if} \quad 0<t<1 \\ 2-t & \text{if} \quad 1<t<2 \\ 0 & \text{otherwise} \end{cases}$$

Hence:

$$|F(t)| \leq 1 \qquad \text{for every t, and}$$

(4.7.3) $$T_3 = An^{2\vartheta + 1/2 + 2\varepsilon} = A_1 N^{4\vartheta + 1 + 4\varepsilon}$$

with $A_1 < A \leq 80$.

4.8. RETURN TO (4.4.3)

We now evaluate

(4.8.1) $$F_1(t) = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T_1(s)\right\}$$

and

(4.8.2) $$F_2(t) = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T_2(s)\right\}$$

4.9. THE EVALUATION OF $F_1(t)$ IN (4.8.1)

According to formula (4.5.2) $F_4(t) = L^{-1}\{T_1(s)\}$ is the sum of a dominant term D(t):

(4.9.1) $$D(t) = 1{,}3203 \prod_{p/t} \frac{p-1}{p-2}$$

which we write as:

(4.9.2) $$D(t) = P(t).\,t$$

where

(4.9.3) $$P(t) = 1{,}3203 \prod_{p/t} \frac{p-1}{p-2}$$

is a discontinuous step function of (t), plus a remainder term R(t), given by:

(4.9.4) $$R(t) = t\,\delta_3\, e^{3\gamma}\ d(t)\frac{(\log\log N)^2}{N}\ \log\log t$$

that contains a discontinuous factor d(t), so that

(4.9.5) $$F_4(t) = D(t) + R(t)$$

Now, the convolution theorem for the inverse Laplace transform states that if

(4.9.6) $$F_3(t) = L^{-1}\{f_3(s)\} \quad \text{and} \quad F_4(t) = L^{-1}\{f_4(s)\}$$

then

(4.9.7) $$L^{-1}\{f_3(s)\, f_4(s)\} = \int_0^t F_3(u) F_4(t-u)\,du$$

In (4.9.6) we choose now

$$f_3(s) = \left(\frac{1-e^{-s}}{s}\right)^2$$

so that F_3 (t) is the F (t) of (4.7.2) and, f_4 (s) = T_1(s) and F_4 (t) = $L^{-1}\{T_1(s)\}$ is the function in (4.9.5)

It follows that

(4.9.8) $$F_1(t) = L^{-1}\{f_3(s)\ f_4(s)\} = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T_1(s)\right\} =$$

$$= \int_0^t F(u)\ F_4(t-u)\,du = \int_0^1 u\,F_4(t-u)\,du + \int_0^1 (2-u)\ F_4(t-u)\,du$$

Now:

(4.9.9) $$\int_0^1 u\,F_4(t-u)\,du = \int_0^1 u\,(t-u)\,du\,.\,P(t-0) + \int_0^1 u\,R(t-u)\,du$$

$$= P(t-0)\left\{\frac{t}{2} - \frac{1}{3}\right\} + \delta_4\, R(t-\delta_5)$$

$0 < \delta_4, \delta_5 < 1.$

(4.9.10) $$\int_1^2 (2-u)F_4(t-u)\,du = \int_1^2 (2-u)(t-u)\,du\,.\,P(t-1+0) + \int_1^2 (2-u)\,R(t-u)\,du$$

$$= P(t-1+0)\left\{\frac{t}{2} - \frac{2}{3}\right\} + \delta_6\, R(t-\delta_7)$$

$1 < \delta_6, \delta_7 < 2$.

Hence:

(4.9.11) $$F_1(t) = P(t-0)\left\{\frac{t}{2}-\frac{1}{3}\right\} + P(t-1+0)\left\{\frac{t}{2}-\frac{2}{3}\right\} + \delta_4 R(t-\delta_5) + \delta_6 R(t-\delta_7)$$

4.10. THE EVALUATION OF $F_2(t)$ IN (4.8.2)

We had:

$$F_2(t) = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T_2(s)\right\}$$

But according to the convolution theorem (4.9.6) – (4.9.7)

$$F_2(t) = \int_0^t F(u) \,.\, L^{-1}\{T_2(s)\} du$$

Where F(u) is the function of (4.7.2) and $L^{-1}\{T_2(s)\}$ was calculated in (4.6.2). We deduce:

(4.10.1) $$F_2(t) = L^{-1}\{T_2(s)\} \int_0^2 F(u)\, du = L^{-1}\{T_2(s)\} =$$

$$= \frac{\delta_2}{4\pi} N^2 (N+1)^2$$

4.11. THE VALUE OF $v(t)$

According to (4.4.3) we had:

$$v_m(t-1) = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 \{T_1(s) + T_2(s) + T_3(s)\}\right\} =$$

$$= F_1(t) + F_2(t) + 2\delta A N^{2\vartheta + 1/2 + \varepsilon}$$

due to (4.9.11), (4.10.1) and (4.7.3)

Hence:

$$v_m(t-1) = P(t-0)\left\{\frac{t}{2}-\frac{1}{3}\right\} + P(t-1+0)\left\{\frac{t}{2}-\frac{2}{3}\right\} + \delta_4 R(t-\delta_5) + \delta_6 R(t-\delta_7)$$

$$+ \frac{\delta_2}{4\pi} N^2 (N+1)^2 + A_1 N^{4\vartheta+1+2\varepsilon}$$

$$= P(t-0)\left\{\frac{t}{2}-\frac{1}{3}\right\} + P(t-1+0)\left\{\frac{t}{2}-\frac{2}{3}\right\} +$$

(4.1.11)
$$\delta_4\delta_3 e^{3\gamma} d(t-\delta_5)\frac{(\log\log N)^2}{N}\log\log(t-\delta_3)+\delta_6\delta_3 e^{3\gamma}\, t\, d(t-\delta_7)\frac{(\log\log N)^2}{N}\log\log(t-\delta_7)$$

$$+\frac{\delta_2}{4\pi}N^2(N+1)^2 + A_1 N^{4\vartheta+1+2\varepsilon}$$

$$=\frac{v(t-1+0)+v(t-1-0)}{2}$$

Replacing t – 1 by t, and equating terms in t ± 0 both sides we deduce:

(4.11.2)
$$v(t) = P(t)(t+1)+\delta_3\ 2\delta_4\ e^{3\gamma}\ t\ d(t)\ \frac{(\log\log N)^2}{N}\log\log(t+1-\delta_5)$$

$$-\frac{2}{3}P(t)+\frac{\delta_2}{2\pi}N^2(N+1)^2 + A_1 N^{4\vartheta+1+2\varepsilon}$$

The terms in P(t) only can be absorbed changing slightly the values of δ_2 and δ_3, so we ignore them in what follows.

As in (4.11.2) we assume P(t) ≠ 0, then, according to (4.5.3) and (4.5.4) we must assume, in what follows, that t is an even number.

4.12. CHOOSING N AS A FUNCTION OF t

We choose now:

(4.12.1) $t = N^5$ $N = t^{\frac{1}{5}}$

Then

(4.12.2)
$$v(t) \approx P(t).t+\delta_6.3{,}56.d(t).(\log\log t)^3 t^{4/5}+\delta_2\frac{80}{2\pi}t^{4/5}+160t^{\frac{48}{5}+\frac{1}{5}+\frac{2\varepsilon}{5}}$$

But we know that $\theta = ½$ and d(t) can be taken as $t^{\frac{\log 2}{\log\log t}}$

Hence

(14.12.3)
$$v(t) > P(t).t+3{,}56.(\log\log t)^3 t^{4/5+\frac{\log 2}{\log\log t}}+\frac{80}{2\pi}t^{4/5}+160t^{\frac{3+2\varepsilon}{5}}$$

This proves unconditionally the truth of the Hardy – Littlewood conjectural formula for the binary Goldbach problem, that $v(t) \sim P(t) \, . \, t$

4.13. MISCELLANEOUS BOUNDS AND MAJORIZATIONS

The minimal value that v(t) can attain if there are solutions is obviously

(4.13.1) $$2 \log^2 t/2$$

and evidently P(t) < 1,3203. Combining (4.12.3) with (4.13.1) we have that there are solutions if

$$1{,}3203\, t - 3{,}56.(\log \log t)^3\, t^{4/5 + \frac{\log 2}{\log \log t}} + 12{,}7 t^{4/5} + 160\, t^{\frac{3+2\varepsilon}{5}} > 2 \log^2 t/2$$

This inequality holds for every $t<10^B$ with 60 < B < 70. On the other hand the conjecture has been verified numerically up to 4×10^{11}.

4.14. IMPROVED LEMMA (3.8.1)

According to ref. [57] we have:

(4.L1) $$\sum_{q>N}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t) = \sum_{d|t} \frac{\mu^2(d)}{\varphi(d)} \sum_{\substack{N|d<q \\ (q,t)=1}} \frac{\mu(q)}{\varphi^2(q)}$$

Now

(4.L2) $$\sum_{\substack{N|d<q \\ (q,t)=1}} \left| \frac{\mu(q)}{\varphi^2(q)} \right| < \sum_{\substack{N|d<q \\ (q,t)=1}} \frac{1}{\varphi^2(q)} < \sum_{N|d<q} \frac{1}{\varphi^2(q)}$$

But for $\varphi_{(n)}$ we have the bound

(4.L3) $$\frac{n}{\varphi(n)} \le e^{\gamma} \log \log n + \frac{5}{2 \log \log n}$$

ref. [21]) valid for every $n \ge 3$ (with one exception), so that

(4.L4) $$\sum_{N|d<q} \frac{1}{\varphi^2(q)} < \sum_{N|d<q} e^{2\gamma} \frac{(\log \log n)^2}{n^2} + \frac{5}{2} \sum_{N|d<q} \frac{1}{n^2 \log \log n}$$

$$< e^{2\gamma} \int_{N|d-1}^{\infty} \frac{(\log \log u)^2}{u^2} du + \frac{5}{2} \int_{N|d-1}^{\infty} \frac{du}{u^2 \log \log u}$$

$$\sim e^{2\gamma} \frac{(\log \log N/d)^2}{(N/d)} + \frac{5}{2} \frac{1}{\log \log N/d \cdot N/d}$$

Replacing (4.L_4) in (4.L_2) we obtain:

(4.L_5)

$$\sum_{\substack{N|d<q \\ (q,t)=1}} \left| \frac{\mu(q)}{\varphi^2(q)} \right| < e^{2\gamma} \frac{(\log \log N/d)^2}{N} \cdot d + \frac{5}{2} \frac{d}{\log \log N/d \cdot N}$$

Replacing (4.L.5) in (4.L.1)

$$\sum_{q>N} \left| \frac{\mu^2(q)}{\varphi^2(q)} C_q(t) \right| < \sum_{d|t} \frac{\mu^2(d)}{\varphi(d)} \left\{ e^{2\gamma} \frac{(\log \log N/d)^2 d}{N} + \frac{5}{2} \frac{d}{N \log \log N/d} \right\}$$

$$< \sum_{d|t} e^{2\gamma} \frac{(\log \log N/d)^2}{N} e^{\gamma} \left(\log \log d + \frac{5}{2 \log \log d} \right)$$

$$< d(t)\, e^{3\gamma} \frac{(\log \log N)^2}{N} \log \log t$$

Lemma (4.14.1). Evaluation of $\sum\sum\sum\sum \frac{\mu(q_1)\mu(q_2)}{\varphi(q_1)\varphi(q_2)} \cdot \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1}$

We have that

$$\frac{1}{A_2 - A_1} = \frac{1}{2\pi i} \frac{1}{\frac{h_2}{q_2} - \frac{h_1}{q_1}} = \frac{q_1 q_2}{2\pi i (h_2 q_1 - h_1 q_2)}$$

Hence

$$\frac{1}{A_2 - A_1} = \frac{q_1 q_2}{2\pi \, |h_2 q_1 - h_1 q_2|} \le \frac{q_1 q_2}{2\pi}$$

because $|h_2 q_1 - h_1 q_2|$ is an integer whose least value is 1.

Furthermore

$$\left| e^{-A_1 t} - e^{-A_2 t} \right| \le 2$$

so that

$$\frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \le \frac{q_1 q_2}{\pi}$$

and

$$\sum_{h_1=0}^{q_1-1} \sum_{h_2=0}^{q_2-1} \left| \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)} \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| \leq \frac{q_1\ q_2}{\pi}$$

because the double sum has $\varphi(q_1)\ \ \varphi(q_2)$ terms.

It follows that

$$\sum_{q_1=1}^{N} \sum_{h_1=0}^{q_1-1} \sum_{q_2=1}^{N} \sum_{h_2=0}^{q_2-1} \left| \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)} \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| <$$

$$< \frac{1}{\pi} \sum_{q_1=1}^{N} \sum_{q_2=1}^{N} q_1\ q_2 = \frac{1}{\pi} \frac{N\,(N+1)}{2} \frac{N(N+1)}{2} = \frac{N^2\,(N+1)^2}{4\,\pi}$$