CHAPTER 6

THE BINARY GOLDBACH PROBLEM

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6.0. INTRODUCTION

As known, the Goldbach problem admits two different statements:

A) The ternary problem: every odd natural number ≥ 7 is the sum of at most three odd primes.

B) The binary problem: every even natural number ≥ 7 is the sum of at most two odd primes.

Such decompositions are called Goldbach decompositions.

Concerning the ternary problem, it has been proved (ref. [13]), that if in the original paper of Hardy-Littlewood (ref. [14]) is assumed the validity of the Extended Riemann Hypothesis (E.R.H.), then the Goldbach decompositions exists for every odd $N > 1.24 \times 10^{50}$.

This bound was lowered by Lucke in 1926 to $N > 10^{32}$, and by D. Zinoviev in 1997 to $N = 10^{20}$.

Finally in 1997 was proved by J.M. Deshouillers that is valid (always under E.R.H.) for every odd $N \geq 7$ (ref. [9]).

Now, very recently, the E.R.H. (together with many others) has been proved in ref. [40] reproduced in Chapter 1 of this book, so that at present the ternary problem can be regarded as completely solved.

Regarding the binary problem, the present status is entirely different. As pointed out by Hardy and Littlewood in their paper above, the circle method is insufficient to prove their conjectural formula even under the assumption of the E.R.H. , and nobody else, in the intervening 80 years already past could surmount the difficulties of the method.

In ref. [10] was proved, under the Riemann hypothesis, that every natural number can be written as a sum of at most four primes; but this is not the binary problem.

According to the six proofs of the Riemann hypothesis given in ref. [40], this is now a new theorem.

Clearly, it was necessary to develop a new method to deal with the binary problem.

Here, the method of the Laplace transform, developed in several papers along 20 years by one of the authors of this result (ref. [28], [35], [38]), has turned out to be the key tool. The results were shown in ref. [2], and we present them in a slightly different form, correcting by the way some few misprints and omissions in it, and improving the bound for which the hypothesis is true to $t > 10^{60}$, which is an interesting bound because it is in the reach of modern computers.

6.1. THE RELATION BETWEEN $\vartheta([u])$ AND $v(u)$

Starting with Chebishev's function $\vartheta(u)$:

(6.1.1) $$\vartheta(u) = \sum_{p \le [u]} \log p$$

where p denotes the prime number, we have:

(6.1.2) $$\Delta\vartheta(u) = \vartheta(u) - \vartheta(u-1) \begin{cases} \log p \text{ if } [u] = p \\ o \text{ in any other case} \end{cases}$$

Hence, if we form

(6.1.3) $$v(t) = \sum_{t = u_1 + u_2} \Delta\vartheta(u_1)\Delta\vartheta(u_2)$$

evidently $v(t)$ is the Hardy – Littlewood function

(6.1.4) $$v(t) = \sum_{p_1 + p_2 = t} \log p_1 \,.\, \log p_2$$

From (6.1.3) we have:

(6.1.5) $$v(t) = \sum_{u_1 < t} \Delta\vartheta(u_1)\ \Delta\vartheta(t - u_1)$$

6.2. TRANSFORMATION OF THE SUM IN (6.1.5) INTO AN INTEGRAL

The $\Delta\vartheta$ (u) function, when is not zero, can be drawn as segments of length 1 at heights log p every time that u = p. More exactly, we have:

(6.2.1) $$\Delta\vartheta(u) = \log p$$

whenever $p \leq u < p + 1$ (otherwise it is zero)

This is shown as shaded bars in fig. (6.1). The graph of ϑ (t-u) is obtained from that of ϑ (u) by rotating the graph about the axis u = t / 2. In fig. (6.3) and (6.4) are shown the cases of

$$y = \vartheta(12 - u) - \vartheta(12 - u - 1)$$

and

$$y = \vartheta(10 - u) - \vartheta(10 - u - 1).$$

Hence the graph of ϑ (u) - ϑ (u - 1) consists of unitary bars at the right of u = p, and that of ϑ (t - u) - ϑ (t - u - 1) consists of unitary bars at the left of u = p.

If we perform now the product

(6.2.2) $$\{\vartheta(u) - \vartheta(u-1)\}\{\vartheta(t-u) - \vartheta(t-u-1)\}$$

It is zero everywhere, except at the points where $u = p_1$, $t - u = p_2$, where it has the value

$$\log p_1 . \log p_2$$

It is evident from the graph that if we wish to obtain this value as an area, we must displace the graph of $\Delta\vartheta$ (u) to the left by one unity, because then:

(6.2.3) $$\int_{p-1}^{p}\{\vartheta(u+1) - \vartheta(u)\}\{\vartheta(t-u) - \vartheta(t-u-1)\} =$$
$$= \log p . \log(t-p)$$

(The shaded areas of log p_1 and log p_2 multiply between themselves). This is shown in fig. (6.2) and (6.5).

The integrand in (6.2.3) vanishes if u < 0 or u > t

Hence, we have that

(6.2.4) $$v(t) = \int_{o}^{t} \Delta\vartheta(u+1)\Delta\vartheta(t-u)\, du$$

This is nothing but the convolution of the two functions that appear at right.

Consequently, if we take the Laplace transform of both sides we obtain, by known rules:

(6.2.5) $$L\{v(t)\} = L\{\Delta\vartheta\,(u+1)\}\,.\,L\{\Delta\vartheta\,(u)\}$$

By the second shift property of the Laplace transform we have:

$$L\{\Delta\vartheta\,(u+1)\} = e^{s}\,L\{\Delta\vartheta\,(u)\}$$

so that (6.2.5) transforms to

(6.2.6) $$L\{v(t)\} = e^{s}\,L^{2}\{\Delta\vartheta\,(u)\}$$

or also:

(6.2.7) $$L\{v(t-1)\} = L^{2}\{\Delta\vartheta\,(u)\}$$

This will be used later.

6.3. THE EVALUATION OF L { Δ ϑ (u)}

From the definition of L, the Laplace transform, we have:

(6.3.1) $$L\{\vartheta(u)\} = \frac{1}{s}\sum_{p}\log p\,.\,e^{-ps}$$

From here follows that

(6.3.2) $$L\{\Delta\vartheta\,(u)\} = \frac{1-e^{-s}}{s}\sum_{p}\log p\,.\,e^{-ps}$$

(p: prime numbers) on account of the shift property

By ref. [25], theorem 248 p. 212, we have:

(6.3.3) $$F(x) = \sum \log p\,.\,x^{p} = \sum_{q=1}^{[\sqrt{n}]}\sum_{\substack{h=1\\(h.q)=1}}^{q}\frac{\mu(q)\,e^{2\pi i h/q}}{\varphi(q)(x-e^{2\pi i h/q})} + An^{\vartheta+1/4+\varepsilon}$$

As a consequence of this we have that

(6.3.4) $$F(s) = \sum \log p\,.\,e^{-ps} = \sum_{q=1}^{[\sqrt{n}]}\sum_{\substack{h=1\\(h.q)=1}}^{q}\frac{\mu(q)}{\varphi(q)(s+2\pi i h/q)} + An^{\vartheta+1/4+\varepsilon}$$

(see ref. [31] for a detailed calculation).

This formula is unconditional, and ϑ denotes the upper bound of the real part of the imaginary zeros of the L – series involved. (Here Re (s) = σ = 1/n: but this restriction does not affect our further reasoning).

Let us analyze formula (6.3.4). The function F (s) has a double infinitude of poles on the line $\sigma = 0$, whenever $s = -2\pi i\, h/q$ where $1\le h \le q$ and $1\le q \le \infty$. They form a natural boundary, and the series at right of (6.3.4) describes the influence of the poles with $q \le [\sqrt{n}]$, while the other term at right accounts for the influence of the poles with $q > [\sqrt{n}]$.

It is evident that the better is the knowledge we have about the zeros of the L-series, the smaller is the remainder term in (6.3.4). According with ref.[40] reproduced in Chapter 1 of this book, we can adopt ϑ = ½ .

6.4. THE USE OF (6.3.4)

Replacement of (6.3.4) in (6.3.2), and of (6.3.2) in (6.2.6) yields:

(6.4.1)
$$L\{v(t)\} = e^s\left(\frac{1-e^{-s}}{s}\right)^2\left\{\sum_{q=1}^{[\sqrt{n}]}\sum_{h=1}^{q}\frac{\mu(q)}{\varphi(q)(s+2\pi i\, h/q)} + An^{\vartheta+1/4+\varepsilon}\right\}^2$$

or

(6.4.2)
$$v_m(t-1) = \frac{v(t-1+0)+v(t-1-0)}{2} = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2\sum_{q}^{[\sqrt{n}]}\sum_{h=1}^{q}\frac{\mu^2(q)}{\varphi^2(q)(s+2\pi i\, h/q)^2}+\right.$$

$$+\left(\frac{1-e^{-s}}{s}\right)^2\sum\sum\sum_{q_1\neq q_2}\sum\frac{\mu(q_1)\mu(q_2)}{\varphi(q_1)\varphi(q_2)(s+2\pi i\; h_1/q_1)(s+2\pi i\; h_2/q_2)}+$$

$$\left.+\left(\frac{1-e^{-s}}{s}\right)^2\sum_{q=1}^{[\sqrt{n}]}\sum_{h}\frac{\mu(q)\left(2An^{\vartheta+1/4+\varepsilon}\right)}{\varphi(q)(s+2\pi i\; h/q)}+\left(\frac{1-e^{-s}}{s}\right)^2 An^{\vartheta+2/4+2\varepsilon}\right\} =$$

$$= L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2\left\{\sum_{q=1}^{[\sqrt{n}]}\sum_{h}\frac{\mu^2(q)}{\varphi^2(q)(s+2\pi i\; h/q)^2}+\right.\right.$$

$$+\sum_{q_1}^{[\sqrt{n}]}\sum_{h_1}\sum_{q_2}^{[\sqrt{n}]}\sum_{h_2}\frac{\mu(q_1)\mu(q_2)}{\varphi(q_1)\varphi(q_2)(s+2\pi i\; h_1/q_1)(s+2\pi i\; h_2/q_2)}+$$

$$+2An^{\vartheta+1/n+\varepsilon}\sum\sum \frac{\mu(q)}{\varphi(q)(s+2\pi i\ h/q)}\Bigg\}+L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 A^2 n^{2\vartheta+1/2+2\varepsilon}\right\}=$$

(6.4.3) $$L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 \{T_1(s)+T_2(s)+T_3(s)\}\right\}+T_3$$

By reasons that will become evident in section 11, we shall evaluate in first place $L^{-1}\{T_1(s)\}$ and $L^{-1}\{T_2(s)\}$.

According to the evaluations performed in ref. [25] we can adopt for A the value

(6.4.4) $A = 80$

(This value however is not relevant for what follows)

6.5. EVALUATION OF $L^{-1}\{T_1(s)\}$

Appealing to tables we have:

(6.5.1) $$L^{-1}\{T_1(s)\} = L^{-1}\left\{\sum_{q=1}^{N}\sum_{h}\frac{\mu^2(q)}{\varphi^2(q)\,(s+2\pi i\, h/q)^2}\right\}$$

$$=\sum_{q=1}^{N}\sum_{h=q}^{q-1}\frac{\mu^2(q)}{\varphi^2(q)}e^{-2\pi i\,ht/q}\,t = \sum_{q=1}^{N}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t)\,.\,t$$

where

$$C_q(t) = \sum_{\substack{h=0\\(h.q)=1}}^{q-1} e^{-2\pi i\,h\,t/q}$$

is Ramanujan's function, and $N = \left[\sqrt{n}\right]$

According to Lemma (6.16.1) we have

$$\left|\sum_{q>n}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t)\right| < d(t)\,e^{3\gamma}\,\frac{(\log\log n)^2}{N}\,\log\log t$$

Hence in (6.5.1) we can put

(6.5.2) $$L^{-1}\{T_1(s)\} = \sum_{q=1}^{\infty}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t)\,.\,t\ +$$

$$+\ \delta_1\,e^{3\gamma}\,d(t)\,\frac{(\log\log N)^2}{N}\,\log\log t\,.\,t$$

Now the singular series

(6.5.3) $$S(t) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t)$$

vanishes for odd t, and for even t can be transformed into an infinite product. (ref. [57]).

(6.5.4) $$S(t) = 2\prod_{p=3}^{\infty}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p|t}\frac{p-1}{p-2}\, t = 1{,}3203 \prod_{p|t}\frac{p-1}{p-2}\,.\, t = P(t)\,.\, t$$

which puts in evidence that S(t) is a discontinuous function.

We have computed thus the influence of the infinitude of double poles in (6.4.2)

6.6. EVALUATION OF $L^{-1}\{T_2(S)\}$

According to tables we have:

(6.6.1) $$F_5(t) = L^{-1}\{T_2(s)\} = L^{-1}\left\{\sum_{q_1=1}^{N}\sum_{h_1}\sum_{q_2=2}^{N}\sum_{h_2}\frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)\,(s + 2\pi i\, h_1/q_1)(s + 2\pi i\, h_1/q_1)}\right\} =$$

$$= \sum_{q_1=1}^{N}\sum_{h_1}\sum_{q_2=2}^{N}\sum_{h_2}\frac{\mu(q_1)\,\mu(q_2)\,(e^{-A_1 t} - e^{-A_2 t})}{\varphi(q_1)\,\varphi(q_2)\,A_2 - A_1}$$

($A_2 = 2\pi i\, h_2/q_2$ $\quad A_1 = 2\pi i\, h_1/q_1$)

Taking into account Lemma (6.16.2), we have:

(6.6.2) $$\left|L^{-1}\{T_2(s)\}\right| = \frac{\delta_2}{4\pi} N^2 (N+1)^2$$

We have computed thus the influence of the simple poles with $q \leq N$ multiplied by themselves.

6.7. THE EVALUATION OF $L^{-1}\{T_3(S)\}$

We have

$$T_3(s) = 2An^{\vartheta+1/4+\varepsilon}\ L^{-1}\sum\sum\frac{\mu(q)}{\varphi(q)(s + 2\pi i\ h/q)} =$$

$$= 2An^{\vartheta+1/4+\varepsilon}\sum_{q}^{N}\frac{\mu(q)}{\varphi(q)}\sum_{h} e^{-2\pi i\, h/q\, t} =$$

$$= 2An^{\vartheta+1/4+\varepsilon}\sum_{q=1}^{N}\frac{\mu(q)}{\varphi(q)} C_q(t)$$

By Lemma (6.16.3), we know that

$$\Delta\vartheta(t) = \Delta\{\vartheta(t) - \vartheta(t-1)\} \sim \sum_{1}^{N} \frac{\mu(q)}{\varphi(q)} C_q(t)$$

Hence

(6.7.1) $$\left|L^{-1}\, T_3(s)\right| \leq 2An^{\vartheta+1/4+\varepsilon} \text{ Log } t = 2AN^{2\vartheta+1/2+\varepsilon} \text{ Log } t$$

6.8. THE EVALUATION OF T_3

We have:

(6.8.1) $$T_3 = An^{2\vartheta + 1/2 + 2\varepsilon} \,.\, L^{-1} \left(\frac{1 - e^{-s}}{s}\right)^2$$

According to tables (ref. [45]), we have:

(6.8.2) $$F(t) = L^{-1} \left(\frac{1 - e^{-s}}{s}\right)^2 = \begin{cases} t & \text{if } 0 < t < 1 \\ 2 - t & \text{if } 1 < t < 2 \\ 0 & \text{otherwise} \end{cases}$$

Hence:

$$|F(t)| \leq 1$$

for every t, and as it zero for t > 2, we can leave aside the influence of T_3.

6.9. RETURN TO (6.4.3)

We must evaluate now

(6.9.1) $$F_1(t) = L^{-1} \left\{\left(\frac{1 - e^{-s}}{s}\right)^2 T_1(s)\right\}$$

and

(6.9.2) $$F_2(t) = L^{-1} \left\{\left(\frac{1 - e^{-s}}{s}\right)^2 T_2(s)\right\}$$

6.10. EVALUATION OF $F_1(T)$ IN (6.9.1)

According to formula (6.5.2) $F_4(t) = L^{-1}\,\{T_1(s)\}$ is the sum of a dominant term D(t):

(6.10.1) $$D(t) = 1{,}3203 \prod_{p|t} \frac{p - 1}{p - 2}\, t$$

which we write as:

(6.10.2) $$D(t) = P(t) . t$$

where

(6.10.3) $$P(t) = 1{,}3203 \prod_{p|t} \frac{p-1}{p-2}$$

is a discontinuous step function of (t), plus a remainder term R(t), given by:

(6.10.4) $$R(t) = t\,\delta_3\, e^{3\gamma}\, d(t) \frac{(\log\log N)^2}{N} \log\log t$$

that contains a discontinuous factor d(t), so that

(6.10.5) $$F_4(t) = D(t) + R(t)$$

Now, the convolution theorem for the inverse Laplace transform states that if

(6.10.6) $$F_3(t) = L^{-1}\{f_3(s)\} \quad \text{and} \quad F_4(t) = L^{-1}\{f_4(s)\}$$

then

(6.10.7) $$L^{-1}\{f_3(s)\, f_4(s)\} = \int_0^t F_3(u)\, F_4(t-u)\, du$$

In (6.10.6) we choose now

$$f_3(s) = \left(\frac{1-e^{-s}}{s}\right)^2$$

so that $F_3(t)$ is the F (t) of (6.8.2) and

$$f_4(s) = T_1(s)$$

then $F_4(t) = L^{-1}\{T_1(s)\}$ is the function in (6.10.5)

It follows that

(6.10.8) $$F_1(t) = L^{-1}\{f_3(s)\, f_4(s)\} = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T_1(s)\right\} =$$

$$= \int_0^t F(u)\, F_4(t-u)\, du = \int_0^1 u\, F_4(t-u)\, du + \int_0^1 (2-u)\, F_4(t-u)\, du$$

Now:

(6.10.9) $$\int_0^1 u\, F_4(t-u)\, du = \int_0^1 u\,(t-u)\, du \,.\, P(t-0) + \int_0^1 u\, R(t-u)\, du$$

$$= P(t-0)\left\{\frac{t}{2} - \frac{1}{3}\right\} + \delta_4\, R(t-\delta_5)$$

$0 < \delta_4, \delta_5 < 1$

(6.10.10) $$\int_1^2 (2-u)\, F_4(t-u)\,du = \int_1^2 (2-u)(t-u)\,du . P(t-1+0) + \int_1^2 (2-u)\, R(t-u)du$$

$$= P(t-1+0) \left\{\frac{t}{2} - \frac{2}{3}\right\} + \delta_6\ R(t-\delta_7)$$

$1 < \delta_6, \delta_7 < 2$

Hence:

(6.10.11) $$F_1(t) = P(t-0)\left\{\frac{t}{2} - \frac{1}{3}\right\} + P(t-1+0)\left\{\frac{t}{2} - \frac{2}{3}\right\} + \delta_4\ R(t-\delta_5) + \delta_6\ R(t-\delta_7)$$

6.11. EVALUATION OF F_2 (t) IN (6.9.2)

We had:

$$F_2(t) = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T_2(s)\right\}$$

But according to the convolution theorem (6.10.6) – (6.10.7)

$$F_2(t) = \int_0^t F(u) . L^{-1} \{T_2(s)\}\ du$$

where F(u) is the function of (6.8.2) and L^{-1} {T_2 (s)} was calculated in (6.6.2).

We deduce:

(6.11.1) $$F_2(t) = L^{-1} \{T_2(s)\} \int_0^2 F(u)\,du = L^{-1}\{T_2(s)\} =$$

$$= \frac{\delta_2}{4\pi} N^2 (N+1)^2$$

6.12. EVALUATION OF $F_5(t) = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 T_3(s)\right\}$

This is a small term and does not need to be evaluated accurately. By the Tauberian theorem of Lemma (6.16.3), this is asymptotically equal to L^{-1} {T_3(s)}, that according to section (6.7.1) is equal to

(6.12.1) $$F_5(t) \sim \delta 2AN^{2\vartheta + 1/2 + \varepsilon}\ \mathrm{Log}\ t$$

6.13. THE VALUE OF v(t)

According to (6.4.3) we had:

$$v_m(t-1) = L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 \{T_1(s) + T_2(s) + T_3(s)\}\right\}$$

$$= F_1(t) + F_2(t) + 2\delta A N^{2\vartheta + 1/2 + \varepsilon} \operatorname{Log} t$$

due to (6.10.11), (6.11.1) and (6.12.1)

Hence:

$$v_m(t\text{-}1) = P(t\text{-}0)\left\{\frac{t}{2} - \frac{1}{3}\right\} + P(t\text{-}1+0)\left\{\frac{t}{2} - \frac{2}{3}\right\} + \delta_4 R(t\text{-}\delta_5) + \delta_6 R(t\text{-}\delta_7)$$

$$+ \frac{\delta_2}{4\pi} N^2 (N+1)^2 + 2\delta A N^{2\vartheta + 1/2 + \varepsilon} \operatorname{Log} t$$

$$= P(t-0)\left\{\frac{t}{2} - \frac{1}{3}\right\} + P(t-1+0)\left\{\frac{t}{2} - \frac{2}{3}\right\} +$$

(6.13.1) $$t\,\delta_4 \delta_3 e^{3\gamma} d(t\text{-}\delta_5) \frac{(\log\log N)^2}{N} \log\log(t-\delta_3) + \delta_6 \delta_3 e^{3\gamma}\, t\, d(t\text{-}\delta_7) \frac{(\log\log N)^2}{N} \log\log(t-\delta_7)$$

$$+ \frac{\delta_2}{4\pi} N^2 (N+1)^2 + 2\delta A N^{2\vartheta + 1/2 + \varepsilon} \operatorname{Log} t$$

$$= \frac{v(t-1+0) + v(t-1-0)}{2}$$

Changing t – 1 by t, and equating terms in $t \pm 0$ both sides we deduce:

(6.13.2) $$v(t) = P(t)(t+1) + \delta_3 2\delta_4 e^{3\gamma} t\, d(t) \frac{(\log\log N)^2}{N} \log\log(t+1\text{-}\delta_5)$$

$$-\frac{2}{3}P(t) + \frac{\delta_2}{2\Pi} N^2 (N+1)^2 + 4\delta A N^{2\vartheta + 1/2 + \varepsilon} \operatorname{Log} t$$

The terms in P(t) only can be absorbed changing slightly the values of δ_2 and δ_3, so we ignore them in what follows.

As in (6.13.2) we assume $P(t) \neq 0$, then, according to (6.5.3) and (6.5.4) we must assume, in what follows, that t is an even number.

6.14. CHOOSING N AS A FUNCTION OF t

We choose now:

(6.14.1) $$t = N^5 \qquad N = t^{\frac{1}{5}}$$

Then

(6.14.2) $$v(t) = P.(t).t + \delta_8\, 2e^{3\gamma}\, d(t) (\log\log t)^3\, t^{4/5}$$

$$+\delta_2 \frac{1}{2\pi} t^{\frac{4}{5}} - 2At^{\frac{2\vartheta}{5}+\frac{1}{10}} \mathrm{Log}\, t$$

The last term is negligible with respect to the preceding ones, and we obtain roughly:

(6.14.3) $$v(t) > P(t).t - 2e^{3\gamma} d(t) (\log\log t)^3 \; t^{4/5} - \frac{1}{2\pi} t^{\frac{4}{5}}$$

(6.14.2) proves unconditionally the truth of the Hardy – Littlewood conjectural formula for the binary Goldbach problem, that

$$v(t) \sim P(t).t$$

6.15. MISCELLANEOUS BOUNDS AND MAJORATIONS

We have evidently, from (6.5.4)

(6.15.1) $$\min D(t) \geq 1{,}3203\, t$$

As concerns d(t) we have:

(6.15.2) $$\limsup_{t\to\infty} d(t) = d_o(t) = t^{\frac{\log 2}{\log\log t}} \qquad \text{(ref. [23])}$$

We shall use in what follows very large values of t, so that we adopt the right hand side as an upper bound for d (t).

Furthermore, the greatest value that v(t) can assume in the case that there is only one solution is $2\log^2 t/2$ (as was indicated in ref. [2]).

Consequently, if we know that

(6.15.3) $$v(t) > 2\log^2 t/2$$

we can be sure that there is at least one solution.

Combining (6.14.2) and (6.15.3), must hold that

(6.15.4) $$v(t) > 1{,}3203\, t - 2e^{3\gamma} d(t).(\log\log t)^3 . t^{4/5} - \frac{1}{\pi} t^{4/5} > 2\log^2 t/2$$

Solving for d (t), this holds roughly if

(6.15.5) $$d(t) < \frac{1{,}3203}{e^{3\gamma}} \frac{t^{1/5}}{(\log\log t)^3}$$

or if

(6.15.6) $$d(t) < 0{,}1168 \frac{t^{1/5}}{(\log\log t)^3}$$

From (6.15.6) and (6.15.2) follows that the Goldbach hypothesis is valid for every value of t such that

(6.15.8) $$d(t) \geq d_o(t)$$

If we adopt the equality sign, we have an equation whose root is

(6.15.9) $$t = t_o \sim 10^B \qquad 50 < B < 60$$

Hence, the binary hypothesis is true for even $t > 10^B$.

For $t < 10^B$, it is valid for every even t that fulfills (6.15.7), that is to say, for the vast majority of them.

In ref. [10] it is proved that the binary Goldbach conjecture is valid up to $t = 10^{14}$, and for the following intervals of t:

$$(10^{15}; 10^{15} + 10^8) \qquad (10^{200}; 10^{200} + 10^9)$$

$$(10^{16}; 10^{16} + 10^8) \qquad (10^{210}; 10^{210} + 10^9)$$

$$\dots \qquad \dots$$

$$(10^{100}; 10^{100} + 10^8) \qquad (10^{300}; 10^{300} + 10^9)$$

The second column now turns out to be superfluous.

6.16. IMPROVED LEMMA 1, LEMMA 2, LEMMA 3

Lemma (6.16.1). According to ref. [57] we have:

(16 L_1) $$\sum_{q>N}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t) = \sum_{d|t} \frac{\mu^2(d)}{\varphi(d)} \sum_{\substack{N|d<q \\ (q,t)=1}} \frac{\mu(q)}{\varphi^2(q)}$$

Now

(16 L_2) $$\sum_{\substack{N|d<q \\ (q,t)=1}} \left| \frac{\mu(q)}{\varphi^2(q)} \right| < \sum_{\substack{N|d<q \\ (q,t)=1}} \frac{1}{\varphi^2(q)} < \sum_{N|d<q} \frac{1}{\varphi^2(q)}$$

But for $\varphi_{(n)}$ we have the bound

(16 L_3) $$\frac{n}{\varphi(n)} \leq e^{\gamma} \log\log n + \frac{5}{2 \log\log n} \qquad \text{(ref. [48])}$$

valid for every $n \geq 3$ (whit one exception), so that

(16 L_4) $$\sum_{N|d<q} \frac{1}{\varphi^2(q)} < \sum_{N|d<q} e^{2\gamma} \frac{(\log\log n)^2}{n^2} + \frac{5}{2} \sum_{N|d<q} \frac{1}{n^2 \log\log n}$$

$$< e^{2\gamma} \int_{N|d-1}^{\infty} \frac{(\log\log u)^2}{u^2}\, du + \frac{5}{2} \int_{N|d-1}^{\infty} \frac{du}{u^2 \log\log u} 1$$

$$\sim e^{2\gamma} \frac{(\log\log N/d)^2}{(N/d)} + \frac{5}{2} \frac{1}{\log\log N/d \cdot N/d}$$

Replacing in (16 L_2) we obtain:

(16 L_5) $$\sum_{\substack{N|d<q \\ (q,t)=1}} \left| \frac{\mu(q)}{\varphi^2(q)} \right| < e^{2\gamma} \frac{(\log\log N/d)^2}{N} \cdot d + \frac{5}{2} \frac{d}{\log\log N/d \cdot N}$$

Replacing in (16 L_1)

$$\sum_{q>N} \left| \frac{\mu^2(q)}{\varphi^2(q)} C_q(t) \right| < \sum_{d|t} \frac{\mu^2(d)}{\varphi(d)} \left\{ e^{2\gamma} \frac{(\log \log N/d)^2 d}{N} + \frac{5}{2} \frac{d}{N \log\log N/d} \right\}$$

$$< \sum_{d|t} e^{2\gamma} \frac{(\log\log N/d)^2}{N} e^{\gamma} \left(\log\log d + \frac{5}{2 \log\log d} \right)$$

$$< d(t)\, e^{3\gamma} \frac{(\log \log N)^2}{N} \log\log t$$

Lemma (6.16.2). Evaluation of $\sum\sum\sum\sum \frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)} \cdot \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1}$

We have that

$$\frac{1}{A_2 - A_1} = \frac{1}{2\pi i} \frac{1}{\frac{h_2}{q_2} - \frac{h_1}{q_1}} = \frac{q_1\ q_2}{2\pi i\,(h_2\ q_1 - h_1\ q_2)}$$

Hence

$$\frac{1}{A_2 - A_1} = \frac{q_1\ q_2}{2\pi\ |h_2\ q_1 - h_1\ q_2|} \le \frac{q_1\ q_2}{2\pi}$$

because $|h_2\, q_1 - h_1\, q_2|$ is an integer whose least value is 1.

Furthermore

$$\left| e^{-A_1 t} - e^{-A_2 t} \right| \leq 2$$

so that

$$\frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \leq \frac{q_1 \; q_2}{\pi}$$

and

$$\sum_{h_1=0}^{q_1-1} \sum_{h_2=0}^{q_2-1} \left| \frac{\mu(q_1)\; \mu(q_2)}{\varphi(q_1)\; \varphi(q_2)} \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| \leq \frac{q_1 \; q_2}{\pi}$$

because the double sum has $\varphi(q_1)\;\;\varphi(q_2)$ terms.

It follows that

$$\sum_{q_1=1}^{N} \sum_{h_1=0}^{q_1-1} \sum_{q_2=1}^{N} \sum_{h_2=0}^{q_2-1} \left| \frac{\mu(q_1)\; \mu(q_2)}{\varphi(q_1)\; \varphi(q_2)} \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} \right| <$$

$$< \frac{1}{\pi} \sum_{q_1=1}^{N} \sum_{q_2=1}^{N} q_1 \; q_2 = \frac{1}{\pi} \frac{N\,(N+1)}{2} \frac{N(N+1)}{2} = \frac{N^2\,(N+1)^2}{4\,\pi}$$

Lemma (6.16.3). The following tauberian theorem can be found in ref.[54] page 366.

Let be F(t) and G(t) two non decreasing functions that fulfill the following condition:

$$\limsup_{\lambda \to \infty} F(\lambda\, t) / F(\lambda) < \infty \qquad \text{for all } t > 0$$

and such that

$$L\{F(u)\} \sim L\{G(u)\} \qquad \text{as } s \to 0$$

Then the exact condition that they must fulfill in order that hold that

$$F(t) \sim G(t) \qquad \text{as } t \to \infty$$

is that

[A] $$\limsup_{\lambda \to \infty} F(\lambda\, t) / F(\lambda)$$

be continuous at $t = 1$.

By (6.3.2) we have:

$$f(s) = L\{\Delta\vartheta(u)\} = \frac{1-e^{-s}}{s}\sum_p \log p \cdot e^{-ps} =$$

$$= \frac{1-e^{-s}}{s}\left\{\sum_{q=1}^{N}\sum_{h=1}^{q}\frac{\mu(q)e^{2\pi i\, h/q}}{\varphi(q)(s+2\pi i\, h/q)} + An^{\vartheta+1/4+\varepsilon}\right\} =$$

$$= \frac{1-e^{-s}}{s}g_N(s) + \frac{1-e^{-s}}{s}An^{\vartheta+1/4+\varepsilon}$$

or

$$\Delta\vartheta(u) = L^{-1}\left\{\frac{1-e^{-s}}{s}g_N(s)\right\} + An^{\vartheta+1/4+\varepsilon}\ L^{-1}\left(\frac{1-e^{-s}}{s}\right)$$

As the second term at right vanishes for $t > 1$, we have:

$$\Delta\vartheta(u) = L^{-1}\left\{\frac{1-e^{-s}}{s}g_N(s)\right\}$$

and

$$L\{\Delta\vartheta(u)\} = \frac{1-e^{-s}}{s}g_N(s)$$

This is asymptotically equal to $g_N(s)$ as $s\to 0$, and besides, $\Delta\vartheta(u)$ fulfills condition [A]. Hence

$$\Delta\vartheta(u) \sim L^{-1}\{g_N(s)\} = \sum_{q=1}^{N}\frac{\mu(q)}{\varphi(q)}C_q(t)$$

This is the result used in section 6.7.

ϑ (u) – ϑ (u – 1)

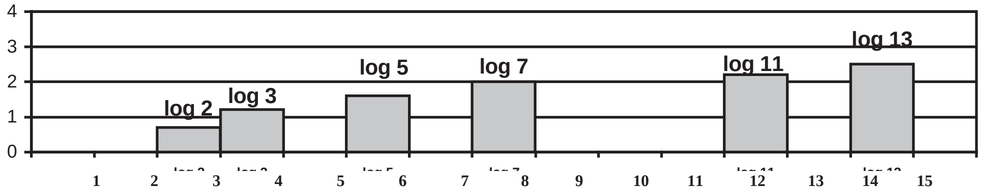

Fig. 6.1

ϑ (u + 1) – ϑ (u)

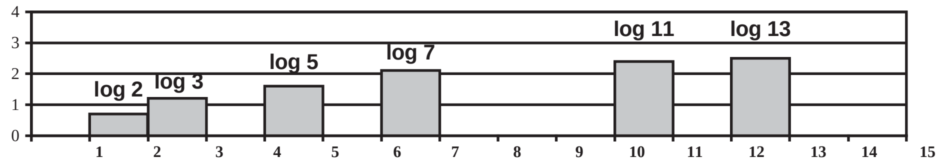

Fig. 6.2

ϑ (12 – u) – ϑ (12 – u – 1)

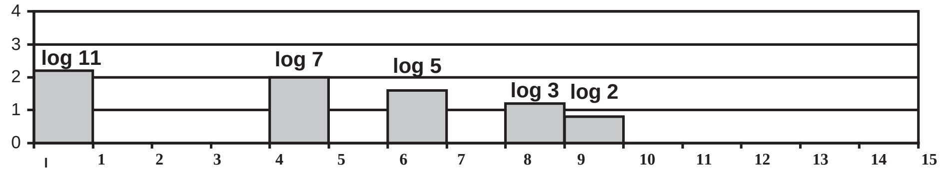

Fig. 6.3

ϑ (10 – u) – ϑ (10 – u – 1)

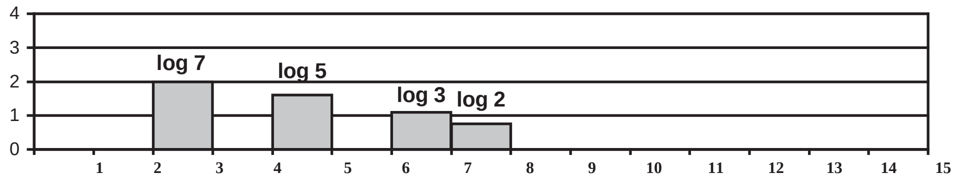

Fig. 6.4

{ϑ (u + 1) – ϑ (u)} {ϑ (12 – u) – ϑ (12 – u – 1)}

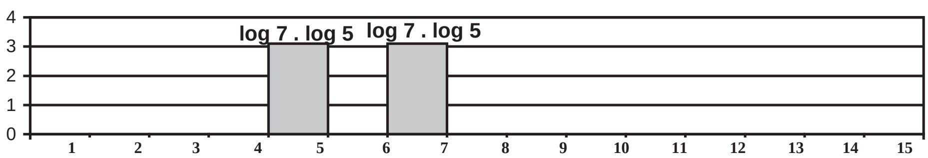

Fig. 6.5

$\{\vartheta\,(u + 1) - \vartheta\,(u)\}\ \ \{\vartheta\,(10 - u) - \vartheta\,(10 - u - 1)\}$

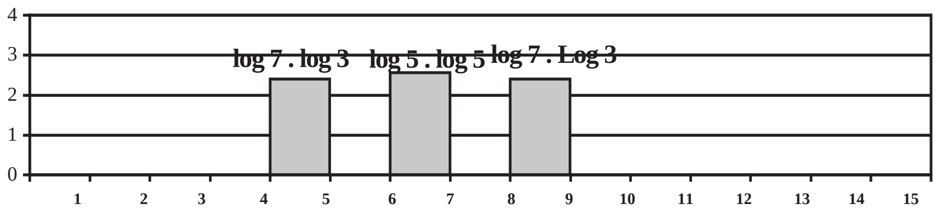

Fig. 6.6