CHAPTER 7

AN ABRIDGED METHOD TO DERIVE THE ASYMPTOTIC FORMULA FOR THE GOLDBACH DECOMPOSITIONS

by Aldo Peretti & Malvina Baica

7.0. INTRODUCTION

In ref.[40] the authors have given a method in order to obtain an exact formula for the Hardy-Littlewood function.

$$v(t) = \sum_{t=p_1+p_2} \log p_1 \,.\, \log p_2$$

Here is indicated how the use of the tauberian theorem quoted in Lemma (6.16.3) of ref.[40] enables us to obtain a much shorter derivation of the asymptotic formula for v(t).

7.1. THE STARTING FORMULA

As was proved in ref.[40], we have that

$$L(v(t)) = e^s \left(\frac{1-e^{-s}}{s}\right)^2 \left\{\sum_q^N \sum_h \frac{\mu(q)}{\varphi(q)(s+2\pi i\, h/q)} + AN^{2\vartheta+1/2+\varepsilon}\right\}^2$$

where L denotes the Laplace transform, ϑ is the upper bound of the real part of the imaginary zeros of the L-series involved, and the formula is valid of $\vartheta \geq 3/4$, which is the actual case.

We write it as

(7.1.1) $$L\{v(t)\} = e^s \left(\frac{1-e^{-s}}{s}\right)^2 \left\{g_N(s) + AN^{2\vartheta+1/2+\varepsilon}\right\}$$

$$= e^s \left(\frac{1-e^{-s}}{s}\right)^2 \left\{g_N{}^2(s) + 2g_N(s)AN^{2\vartheta+1/2+\varepsilon} + A^2N^{4\vartheta+1+2\varepsilon}\right\}$$

Due to the fact that

$$L^{-1}\left\{e^s\left(\frac{1-e^{-s}}{s}\right)^2\right\}=\begin{cases} t & \text{if } 0<t<1 \\ 2-t & \text{if } 1<t<2 \\ 0 & \text{otherwise}\end{cases}$$

the term at the right hand side of (7.1.1) has not any relevance for v(t), and we can put

$$L\{v(t)\}=e^s\left(\frac{1-e^{-s}}{s}\right)^2\left\{g_N{}^2(s)+2g_N(s)AN^{2\vartheta+1/2+\varepsilon}\right\}$$

if t > 2.

By the tauberian theorem of Lemma (6.16.3) of Chapter 6

(7.1.2) $$v(t)\sim L^{-1}\left\{g_N{}^2(s)+2AN^{2\vartheta+1/2+\varepsilon}\,g_N(s)\right\}=$$

$$=L^{-1}\left\{\sum\sum\frac{\mu^2(q)}{\varphi^2(q)(s+2\pi i\,h/q)}+\sum\sum_{q_1\neq q_2}\sum\sum\frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\varphi(q_2)(s+2\pi i\,h_1/q_1)(s+2\pi i\,h_2/q_2)}\right.$$

$$\left.+2AN^{2\vartheta+1/2+\varepsilon}\sum\sum\frac{\mu(q)}{\varphi(q)(s+2\pi i\,h/q)}\right\}=$$

$$=\sum_{q=1}^{N}\frac{\mu^2(q)}{\varphi^2(q)}Cq(t)\cdot t+\sum\sum\sum\sum\frac{\mu(q_1)\,\mu(q_2)\,e^{-A_1t}-e^{-A_2t}}{\varphi(q_1)\,\varphi(q_2)\,A_2-A_1}+$$

$$+2AN^{2\vartheta+1/2+\varepsilon}\sum_{q=1}^{N}\frac{\mu(q)}{\varphi(q)}C_q(t)$$

But

$$\sum_{q=1}^{N}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t).t=\sum_{q=1}^{\infty}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t).t+\delta_1 e^{3\gamma}d(t)\frac{(\log\log N)^2}{N}\log\log t.t$$

(7.1.3) $$\left|\sum\sum\sum\sum\frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)}\frac{e^{-A_1t}-e^{-A_2t}}{A_2-A_1}\right|\leq\frac{\delta_2}{2\pi}N^2(N+1)^2$$

$$\vartheta(t)-\vartheta(t-1)\sim\sum_{q=1}^{N}\frac{\mu(q)}{\varphi(q)}C_q(t)$$

(Where $\vartheta(t)$ is the Chebishev function $\vartheta(t)=\sum_{p\leq t}\log p$)

as was shown in ref.[40] by Lemmas (6.16.1),(6.16.2) and (6.16.3) of Chapter 6 in this book.

Hence

$$v(t) \sim \sum_{1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t) \cdot t + \delta_1 \, t \, e^{3\gamma} \, d(t) \frac{(\log \log N)^2}{N} \log \log t + \frac{\delta_2}{2\pi} N^2 (N+1)^2 + \delta_3 2AN^{2\vartheta+1/2+\varepsilon} \log t \tag{7.1.4}$$

We choose now $t = N^5$, so that

$$v(t) \sim \sum_{1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t) \cdot t + \delta_1 \, e^{3\gamma} \, d(t) (\log \log t)^3 . \, t^{4/5} + \frac{\delta_2}{2\pi} t^{4/5} + \delta_3 \, 2 \, A \, t^{\frac{2}{5}\vartheta + \frac{1}{10} + \varepsilon} \log t \tag{7.1.5}$$

It is evident now the little influence that the value of ϑ has upon the value of v(t).

The preceding formula coincides, in its essential features, with that deduced by the exact method.

Due to the multiplicative properties of $\mu(q)$, $\varphi(q)$ and $C_q(t)$ the series in the first term at right can be written as:

$$\sum_{1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t) = 2 \prod_{p=3}^{\infty} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|t} \frac{p-1}{p-2} = 1{,}3203 \prod_{p|t} \frac{p-1}{p-2} \tag{7.1.6}$$

So that (7.1.5) turns out to be

$$v(t) \sim 1{,}3203 \prod_{p|t} \frac{p-1}{p-2} . \, t + 0(t^{\frac{4}{5}+\varepsilon})$$

From (7.1.5) follows, as was shown in ref.[1], that the Goldbach hypothesis is correct for even $t > 10^{60}$.