CHAPTER 8

UNCONDITIONAL SOLUTION OF THE TWIN PRIME PROBLEM

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8.0. INTRODUCTION

The twin prime conjecture asserts that there are infinitely many couples of primes whose difference is a fixed number k. For instance, when k=2 we have the examples

$$5,7 \ ; \ 11,13 \ ; \ 17,19 \ ; \ \ldots 101,103 \ldots \ ; \ 33218925.2^{169.690} \pm 1 \ldots$$

(this last pair has 51090 digits)

In 1920, Viggo Brun, using a sieve method (ref. [6]), proved that if we denote with N(x) the quantity of such pairs that are $\leq x$, then

(8.0.1) $$N(x) \leq \frac{c\,x}{\log^2 x} \qquad \text{if} \quad k=2$$

This proves that the sum of the inverses of the couples is convergent, but does not prove the existence of infinitely many twin primes. Of course, since that date, Brun´s result has been considerably improved as regards the value of the constant c.

In their most famous 1923 paper (ref. [14]), Hardy and Littlewood conjectured an asymptotic formula for $P_k(x)$, the quantity of twin primes with difference k, that are $\leq x$.

Their way for deriving it was not clear at all; but the present author, in ref. [37], has reconstructed it in detail, showing that they performed three hypotheses before arriving to their final formula. Very likely this is why at present the problem is currently considered as hopeless.

8.1 AN INTERSECTION FORMULA

Hardy-Littlewood did not work directly with the function N(x), but they evaluated in change

(8.1.1) $$S(n) = \sum_{twins \le n} \log p_1 \cdot \log p_2$$

Now, if

(8.1.2) $$\vartheta(x) = \sum_{p \le x} \log p \qquad \text{(p: prime number)}$$

is the Chebishev function, then evidently

(8.1.3) $$\Delta\vartheta(u) = \vartheta(u) - \vartheta(u-1) = \begin{cases} \log u & \text{if } u = \text{prim number } p_i \\ 0 & \text{in any other case} \end{cases}$$

and

(8.1.4) $$\Delta\vartheta(u-k) = \vartheta(u-k) - \vartheta(u-k-1) = \begin{cases} \log(u-k) & \text{if } u-k = p_j \\ 0 & \text{in any other case} \end{cases}$$

so that

(8.1.5) $$\{\vartheta(u) - \vartheta(u-1)\}\{\vartheta(u-k) - \vartheta(u-k-1)\} = \begin{cases} \log p_i \, . \, \log p_j & \text{with } p_i - p_j = k \\ 0 & \text{in any other case} \end{cases}$$

It follows that

(8.1.6) $$S(t) = \int_o^t \Delta\vartheta(u) \, . \, \Delta\vartheta(u-k) \, . \, du$$

8.2 THE ASYMPTOTIC VALUE OF $\Delta\vartheta$ (u)

In Chapter 6, Lemma (6.16.3) of this book it is derived the following asymptotic formula for $\Delta\vartheta$ (u):

(8.2.1) $$\Delta\vartheta(u) \sim \sum_{q=1}^{N} \frac{\mu(q)}{\varphi(q)} Cq(t)$$

where $\mu(q)$ is the Moebius function, $\varphi(q)$ the Euler function and $C_q(t)$ the Ramanujan function

(8.2.2) $$C_q(t) = \sum_{\substack{h=1 \\ (h.q)=1}}^{q} e^{-2\pi i \frac{h}{q} t}$$

As known, $C_q(u)$ is a real and even function of u, because we have:

(8.2.3) $$C_q(u) = \sum_{\substack{h=1 \\ (h.q)=1}}^{h=q} \cos 2\pi \frac{h}{q} u$$

8.3 RETURN TO (8.1.6)

From (8.2.1) we deduce:

(8.3.1) $$\Delta\vartheta(u-k) \sim \sum_{q=1}^{N} \frac{\mu(q)}{\varphi(q)} C_q(u-k)$$

This can be written also (because $C_q(u)$ is an even function of u) as:

(8.3.2) $$\Delta\vartheta(u-k) \sim \sum_{1}^{N} \frac{\mu(q)}{\varphi(q)} C_q(\kappa-u)$$

Replacing (8.2.1) and (8.3.2) in (8.1.6) we obtain an asymptotic approximation to S(t):

(8.3.3) $$S(t) \approx \int_{o}^{t} \left\{ \sum_{1}^{N} \frac{\mu(q_1)}{\varphi(q_1)} C_{q_1}(u) \right\} \left\{ \sum_{1}^{N} \frac{\mu(q_2)}{\varphi(q_2)} Cq_2(k-u) \right\} du$$

$$= \int_{o}^{t} \left\{ \sum_{1}^{N} \frac{\mu^2(q)}{\varphi^2(q)} C_q(k) + \sum_{q_1} \sum_{\substack{q_2 \\ q_1 \neq q_2}} \frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)} C_{q_1}(u)\, C_{q_2}(k-u) \right\} du$$

because, from its definition (8.2.2) follows that

(8.3.4) $$C_q(u)\, C_q(k-u) = C_q(k)$$

From (8.3.3)

$$S(t) \sim \sum_{1}^{N} \frac{\mu^2(q)}{\varphi^2(q)} C_q(k).t + \int_{o}^{t} \sum\sum \frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)} C_{q_1}(u) C_{q_2}(k-u)\, du$$

In Chapter 6 of this book, Lemma (6.16.1) is proved that

$$\left| \sum_{q>N}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(k).t \right| < d(k)\, e^{3\gamma} \frac{(\log\log N)^2}{N} \log\log k.t$$

where d(k) is the quantity of divisors of k, and γ = Euler's constant.

Hence

(8.3.5) $$S(t) \sim \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(k).t + \int_{o}^{t} \sum_{q_1=1}^{N} \sum_{q_2=1}^{N} \frac{\mu(q_1)\mu(q_2)}{\varphi(q_1)\varphi(q_2)} C_{q_1}(u)\, C_{q_2}(k-u)\, du$$

$$+ \delta_1 d(k)\, e^{3\gamma} . \frac{(\log\log N)^2}{N} \log\log k\,.\,t \qquad (|\delta| < 1)$$

The multiplicative properties of the terms in the first series at right allows us to transform it into a product, and we have:

$$\sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t).t = 2\prod_{p=3}^{\infty}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|k}\frac{p-1}{p-2}.t$$

$$= 1{,}3023 \prod_{p|k}\frac{p-1}{p-2}.t$$

8.4. EVALUATION OF THE SECOND TERM OF (8.3.5)

We have

$$\int_0^t C_{q_1}(u)\,C_{q_2}(k-u)\,du = \int_0^t \sum_{h_1=1}^{q_1} e^{-2\pi i \frac{h_1}{q_1}u} \sum_{h_2=1}^{q_2} e^{-2\pi i \frac{h_2}{q_2}(k-u)}\,du$$

$$= \sum_{h_1}\sum_{h_2} e^{-2\pi i \frac{h_2}{q_2}k} \int_0^t e^{-2\pi i\left(\frac{h_1}{q_1}-\frac{h_2}{q_2}\right)u}\,du =$$

$$= \sum_{h_1}\sum_{h_2} e^{-2\pi i \frac{h_2}{q_2}k} \frac{1-e^{-At}}{2\pi i\left(\frac{h_2}{q_2}-\frac{h_1}{q_1}\right)} \qquad \left(A = 2\pi i\left(\frac{h_1}{q_1}-\frac{h_2}{q_2}\right)\right)$$

$$= \sum_{h_1}\sum_{h_2} e^{-2\pi i \frac{h_2}{q_2}k} \frac{q_1 q_2 (1-e^{-At})}{2\pi i(q_1 h_2 - h_1 q_2)}$$

As $q_1 \neq q_2$ the smallest value of $|q_1h_2 - h_1q_2|$ is 1, while the modulus of the numerator is less or equal than $2q_1\, q_2$.

The quantity of terms in the sum is $\varphi(q_1)\varphi(q_2)$. Hence the modulus of the double sum does not exceed of

$$\frac{1}{\pi} q_1 \varphi_1 q_2 \varphi_2$$

Returning to (8.3.5), we have:

(8.4.1)
$$\left|\int_0^t \sum_{q_1=1}^{N}\sum_{q_2=1}^{N} \frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)} C_{q_1}(u)\,C_{q_2}(k-u).du\right| <$$

$$< \sum_{q_1=1}^{N}\sum_{q_2=1}^{N} \frac{q_1\,q_2}{\pi} = \frac{N^2(N+1)^2}{4\pi}$$

Replacement of this value in (8.3.5) yields:

(8.4.2)
$$S(t) \sim 1{,}3023\prod_{p|k}\frac{p-1}{p-2}t + \delta_0\frac{N^2(N+1)^2}{4\pi} + \\ + \delta_1 d(k)e^{3\gamma}\frac{(\log\log N)^2}{N}\log\log k.t$$

8.5. CHOICE OF N AS A FUNCTION OF t

In order to make the two error terms in (8.4.2) approximately equal we choose $N=t^{1/5}$. We get:

$$S(t) \sim 1{,}3023\prod_{p|k}\frac{p-1}{p-2}t + \delta_0 t^{4/5} + \delta_1 d(k)\, e^{3\gamma}(\log\log t)^2.t^{4/5}\log\log k$$

As $d(k) = 0(k^{\varepsilon})$ and $\log\log k = 0(k^{\varepsilon})$, it can be written also as

(8.5.1)
$$S(t) \sim 1{,}3023\prod_{p|k}\frac{p-1}{p-2}t + 0(k^{\varepsilon}\ t^{4/5+\varepsilon})$$

This formula proves:

A) That there are infinitely many pairs of twin primes

B) That the Hardy-Littlewood hypothesis that S(t) is approximated by the first term at right of (8.5.1) is correct.

Concerning $P_k(t)$, the quantity of twin primes with difference κ equal or less than t. Hardy and Littlewood proved that:

$$P_k(t) \sim \frac{S(t)}{\log^2 t}$$

so that from (8.5.1) follows:

$$P_k(t) \sim 1{,}3023\prod_{p|k}\frac{p-1}{p-2}\frac{t}{\log^2 t} + 0(k^{\varepsilon}\ \frac{t^{4/5+\varepsilon}}{\log^2 t})$$