CHAPTER 9

A BRIEF AND COMPLETE SOLUTION OF THE CUBOIDS PROBLEM

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9.0. INTRODUCTION

The cuboid problem goes back to Euler, who asked to find those parallelpipeds with integral edges and integral diagonals in their faces.

Analytically, this means to solve the following system of equations:

(9.0.1)
$$\begin{cases} x^2 + y^2 = w^2 \\ x^2 + z^2 = v^2 \\ y^2 + z^2 = u^2 \end{cases}$$

The minimal solution has been found by computers, and is:

$$44^2 + 117^2 = 125^2$$
$$44^2 + 240^2 = 244^2$$
$$117^2 + 240^2 = 267^2$$

ref.[40]

9.1. INTERSECTION FORMULAS

We have:

(9.1.1)
$$\Delta\{[\sqrt{t}]\} = [\sqrt{t}] - [\sqrt{t-1}] = \begin{cases} 1 \text{ if } n^2 \le t < n^2 + 1 \quad (n = \text{integer}) \\ 0 \text{ in any other case} \end{cases}$$

Otherwise stated, the graph of $\Delta\{[\sqrt{t}]\}$ consists of segments of length 1 at height 1, placed at the right hand side of the square numbers. Below each segment there is an area of value 1.

Here [u] denotes the greatest integer function.

Hence:

(9.1.2)
$$\left[\sqrt{x^2 + y^2}\right] - \left[\sqrt{x^2 + y^2 - 1}\right] = \begin{cases} 1 \text{ if } m^2 \le x^2 + y^2 < m^2 + 1 \\ 0 \text{ in other case} \end{cases}$$

$$\left[\sqrt{x^2+y^2}\right]-\left[\sqrt{x^2+z^2-1}\right]=\begin{cases}1 \text{ if } p^2 \le x^2+z^2 < p^2+1\\ 0 \text{ in other case}\end{cases}$$

$$\left[\sqrt{y^2+z^2}\right]-\left[\sqrt{x^2+z^2-1}\right]=\begin{cases}1 \text{ if } q^2 \le y^2+z^2 < q^2+1\\ 0 \text{ in other case}\end{cases}$$

It is evident then that the quantity of solutions N(x_0 , y_0 , z_0) of system (9.0.1) with $x \le x_0$, $y \le y_0$, $z \le z_0$ is given by the expression:

(9.1.3) $$N(x_0,y_0,z_0)=\sum_{x_1=1}^{x_0}\sum_{y_1=1}^{y_0}\sum_{z_1=1}^{z_0}\left\{\left[\sqrt{x^2+y^2}\right]-\left[\sqrt{x^2+y^2-1}\right]\right\}\times \times\left\{\left[\sqrt{x^2+z^2}\right]-\left[\sqrt{x^2+z^2-1}\right]\right\}\times\left\{\left[\sqrt{y^2+z^2}\right]-\left[\sqrt{y^2+z^2-1}\right]\right\}$$

For the sake of simplicity we write this as:

(9.1.4) $$N(x_0,y_0,z_0)=\sum\sum\sum\Delta\left\{\left[\sqrt{x^2+y^2}\right]\right\}\cdot\Delta\left\{\left[\sqrt{x^2+z^2}\right]\right\}\cdot\Delta\left\{\left[\sqrt{y^2+z^2}\right]\right\}$$

Taking into account the values of the Δ's as given in (2.2), we deduce:

(9.1.5) $$N(x_0,y_0,z_0)\cong\int_1^{x_0}\int_1^{y_0}\int_1^{z_0}\Delta\left\{\left[\sqrt{x^2+y^2}\right]\right\}\cdot\Delta\left\{\left[\sqrt{x^2+z^2}\right]\right\}\cdot\Delta\left\{\left[\sqrt{y^2+z^2}\right]\right\}\cdot dx\cdot dy\cdot dz$$

The equality sign holds when x_0 , y_0 and z_0 fall in an interval without squares or squares + 1. In contrary case holds the asymptotic sign ~.

9.2. LEMMA (9.2.1)

The following theorem can be found in ref [54].

Let be F(t) and G(t) two non decreasing functions that fulfill the following condition:

$$\text{lim sup } F(\lambda t)/F(\lambda)<\infty \quad \text{for all } t>0$$

and such that

$$L\{F(u)\}\approx L\{G(u)\} \quad \text{as } s\to 0$$

Then the exact condition that they must fulfill in order that hold that:

$$F(t) \approx G(t) \qquad \text{as } t \to \infty$$

is that

$$\limsup_{\lambda \to \infty} F(\lambda t)/F(\lambda) \text{ be continuous at t=1}$$

In the case of the function:

$$F(t) = \Delta\{[\sqrt[k]{t}]\}$$

we must check if:

$$\lim_{\lambda\to\infty} \frac{F(\lambda t)}{F(\lambda)} = \lim_{\lambda\to\infty} \frac{[\sqrt[k]{\lambda t}] - [\sqrt[k]{(\lambda-1)t}]}{[\sqrt[k]{\lambda}] - [\sqrt[k]{\lambda-1}]} = \lim_{\lambda\to\infty} \frac{[\sqrt[k]{\lambda(1+\varepsilon)}] - [\sqrt[k]{(\lambda-1)(1+\varepsilon)}]}{[\sqrt[k]{\lambda}] - [\sqrt[k]{\lambda-1}]}$$

is continuous when $\varepsilon \to 0$. Indeed, this is the case.

9.3. THE ASYMPTOTIC VALUE OF $\Delta\{[\sqrt[k]{t}]\}$

The formula

(9.3.1) $$L\{[\sqrt[k]{t}]\} = \frac{1}{s}\sum_{n=1}^{\infty} e^{-sn^k}$$

can be found in tables, or easily deduced from the definition of L, the Laplace transform.

From this follows (by the translation property of the Laplace transform) that:

(9.3.2) $$L\{[\sqrt[k]{t}] - [\sqrt[k]{t-1}]\} = \frac{1-e^{-s}}{s}\sum_{n=1}^{\infty} e^{-sn^k} = f(s)$$

The function represented by the above series has a natural boundary on $s = \sigma = 0$

Its Farey dissection

(9.3.3) $$\sum_{n=1}^{\infty} e^{-sn^k} = \Gamma(1+1/k)\sum_{q=1}^{N}\sum_{h=0}^{q-1} \frac{w(k,q,h)}{q(s+2\pi i\frac{h}{q})^{1/k}} + O(N^{1/k-1})$$

(h.q) = 1

shows the poles it has there, and the values of the respective residues. Here

$$w(k,q,h) = \sum_{x=0}^{q-1} e^{2\pi i x^k h/q}$$

is the Weyl sum. φ∂

From (9.3.2) and (9.3.3), putting that $\Delta\left[\sqrt[k]{t}\right]=\left[\sqrt[k]{t}\right]-\left[\sqrt[k]{t-1}\right]$, follows:

(9.3.4) $$f(s)=L\left\{\Delta\left[\sqrt[k]{t}\right]\right\}=\frac{1-e^{-s}}{s}\left\{T(1+1/k)\sum_{q=1}^{N}\sum_{h=0}^{q-1}\frac{w(k,q,h)}{q(s+2\pi ih/q)^{1/k}}+O(N^{1/k-1})\right\}$$

or

(9.3.5) $$\Delta\left[\sqrt[k]{t}\right]=L^{-1}\left\{\frac{1-e^{-s}}{s}g_N(s)+\frac{1-e^{-s}}{s}O(N^{1/k-1})\right\}$$

where L^{-1} denotes the inverse Laplace transform.

The last term at right is entirely negligible. In fact, we have according to tables, that:

$$L^{-1}\left\{\frac{1-e^{-s}}{s}\right\}=\begin{cases}1 \text{ if } 0<t<1\\ 0 \text{ if } t>1\end{cases}$$

As we are interested only in the case when t > 1, we write (9.3.5) as:

(9.3.6) $$\Delta\left[\sqrt[k]{t}\right]=L^{-1}\left\{\frac{1-e^{-s}}{s}g_N(s)\right\}=L^{-1}\left\{f^*(s)\right\}=f(t)$$

we compare now $f^*(s)$ with

(9.3.7) $$g^*(s)=L^{-1}\{g_N(s)\}$$

It is evident that

(9.3.8) $$f^*(s)\approx g^*(s)$$

as $s\to 0$.

The function $\Delta\left[\sqrt[k]{t}\right]$ fulfills the requirements of the Tauberian theorem of Lemma (9.2.1), and so we can assert that:

(9.3.9) $$F(t)\approx G(t) \text{ as } t\to\infty$$

where

(9.3.10) $$f(t)=L^{-1}\left\{f^*(s)\right\} \qquad G(t)=L^{-1}\left\{g^*(s)\right\}$$

Hence

(9.3.11) $$F(t)\approx G(t)=L^{-1}\{g_N(s)\}=L^{-1}\left\{T(1+1/k)\sum_{q=1}^{N}\sum_{h=0}^{q-1}\frac{w(k,q,h)}{q(s+2\pi ih/q)^{1/k}}\right\}=$$

$$=\frac{t^{1/k-1}}{k}\sum_{(h,q)=1}\sum\frac{w(k,q,h)}{q}e^{-2\pi ih/qt}$$

From (9.3.10) and (9.3.6) follows our final formula:

(9.3.12) $$\Delta\left[\sqrt[k]{t}\right] \approx \frac{t^{1/k-1}}{k} \sum_{q=1}^{N} \sum_{\substack{h=0 \\ (h,q)=1}}^{q-1} \frac{w(k,q,h)}{q} e^{-2\pi i t h/q}$$

9.4. RETURN TO THE TRIPLE INTEGRAL

Now we consider the special case k=2 in the preceding formula. We obtain:

(9.4.1) $$\Delta\left(\left[\sqrt{t}\right]\right) \approx \frac{1}{2} \sum_{q} \left(\sum_{h} \frac{w(2,q,h)}{q} e^{-2\pi i t h/q} \right) t^{-1/2}$$

which we write simply as:

$$\Delta\left(\left[\sqrt{t}\right]\right) \approx \frac{1}{2} \sum_{q=1}^{N} \frac{s(q,t)}{q} t^{-1/2}$$

it follows that

$$\Delta\left(\left[\sqrt{x^2+y^2}\right]\right) \approx \frac{1}{2} \sum_{q=1}^{N} \frac{s(q,t_1)}{q} (x^2+y^2)^{-1/2}$$

(9.4.2) $$\Delta\left(\left[\sqrt{x^2+z^2}\right]\right) \approx \frac{1}{2} \sum_{q=1}^{N} \frac{s(q,t_2)}{q} (x^2+z^2)^{-1/2}$$

$$\Delta\left(\left[\sqrt{x^2+z^2}\right]\right) \approx \frac{1}{2} \sum_{q=1}^{N} \frac{s(q,t_3)}{q} (y^2+z^2)^{-1/2}$$

with $t_1 = x^2 + y^2 \quad t_2 = x^2 + z^2 \quad t_3 = y^2 + z^2$

introduction of these formulas in (9.1.5) yields

(9.4.3) $$N(x_0, y_0, z_0) \approx \int_1^{x_0} \int_1^{y_0} \int_1^{z_0} \frac{1}{8} \left(\sum \frac{S(q_1 t_1)}{q} \right) \left(\sum \frac{S(q_1 t_2)}{q} \right) \left(\sum \frac{S(q_1 t_3)}{q} \right) \cdot \frac{dx.dy.dz}{\sqrt{(x^2+y^2)(x^2+z^2)(y^2+z^2)}}$$

we apply now the first mean value theorem of the integral calculus successively to the variables, and deduce:

(9.4.4) $$N(x_0, y_0, z_0) \approx \frac{i}{8}\sum\frac{S(q,\xi_1)}{q}\sum\frac{S(q,\xi_2)}{q}\sum\frac{S(q,\xi_3)}{q}$$

$$\int_1^{x_0}\int_1^{y_0}\int_1^{z_0}\frac{dx.dy.dz}{\sqrt{(x^2+y^2)(x^2+z^2)(y^2+z^2)}}$$

where

$\xi_1 = x_1^2 + y_1^2$ with $1<x_1<x_0$ $1<y_1<y_0$ and so on

The question is now the evaluation of the integral

$$I_1 = \int_1^{x_0}\int_1^{y_0}\int_1^{z_0}\frac{dx.dy.dz}{\sqrt{(x^2+y^2)(x^2+z^2)(y^2+z^2)}}$$

In order to pass from Cartesian coordinates (x, y, z) to spherical coordinates (r, φ, ϑ) holds the formula

$$\iiint_V f(x,y,z)\,dx\cdot dy\cdot dz = \iiint_V f(r\cdot\sin\varphi\cdot\cos\vartheta, r\cdot\sin\varphi\cdot\sin\vartheta, r\cdot\cos\varphi)\,r^2\sin\varphi\cdot dr\cdot d\vartheta\cdot d\varphi$$

see ref [8]

If $1 \le x \le x_0$, $1 \le y \le y_0$, $1 \le z \le z_0$, then:

$$x^2 + y^2 + z^2 \le x_0^2 + y_0^2 + z_0^2$$

This last value is nothing but D_0^2, the square of the diagonal of the cuboid.

Then r varies from $\sqrt{3}$ to D_0; the angle φ varies between 0 and

$$\varphi_0 \le \frac{\pi}{2}$$

the angle ϑ_0 varies between ϑ_1 and, consequently we have:

$$\vartheta_0 \le \frac{\pi}{2}$$

(9.4.5) $$I_1 = \frac{1}{8}\cdot\int_{\sqrt{3}}^{D_0}\frac{dr}{r}\int_0^{\varphi_0}\frac{d\varphi}{\varphi}\int_{\vartheta_1}^{\vartheta_0}\frac{d\vartheta}{\vartheta}\cdot\frac{1}{\sqrt{(\sin^2\varphi\cdot\cos^2\vartheta+\cos^2\varphi)(\sin^2\varphi\cdot\sin^2\vartheta+\cos^2\varphi)}}$$

The integrals involving φ and ϑ are independent of r, and can be represented by a certain constant $C(\varphi_0,\vartheta_0)$. Hence we deduce:

(9.4.6) $$I_1 = \frac{1}{8}C(\varphi_0,\vartheta_0)\log D_0 + O(1)$$

Replacing in (9.4.5) we obtain:

(9.4.7) $$N(x_0, y_0, z_0) \approx \frac{S}{8} C(\varphi_0, \vartheta_0) \log D_0 + O(1)$$

A first consequence of (9.4.7) is that there are infinitely many solutions of the system (9.0.1).

Many of them have been determined by actual calculation or still, by parametric formulas (see ref. [5], [7], [26] and [53]).

9.5. PERFECT CUBOIDS

What happens in the problem of cuboids when it is required that also the diagonal $D = \left(\sqrt{x^2 + y^2 + z^2}\right)$ be an integer? (The so called "perfect cuboids"). This is equivalent to solve the system:

(9.5.1) $$\begin{cases} x^2 + y^2 = u^2 \\ x^2 + z^2 = v^2 \\ y^2 + z^2 = u^2 \\ x^2 + y^2 + z^2 = D^2 \end{cases}$$

The quantity of solutions with $1 \le x \le x_0$, $1 \le y \le y_0$, $1 \le z \le z_0$ is now:

$$N(x_0, y_0, z_0) = \sum_1^{x_0}\sum_1^{y_0}\sum_1^{z_0} \Delta\left\{\sqrt{x^2 + y^2}\right\}\Delta\left\{\sqrt{x^2 + z^2}\right\}\Delta\left\{\sqrt{y^2 + z^2}\right\}\Delta\left\{\sqrt{x^2 + y^2 + z^2}\right\}$$

It follows that:

$$N(x_0, y_0, z_0) = \frac{S_1}{2^4} \int_1^{x_0}\int_1^{y_0}\int_1^{z_0} \frac{dx \cdot dy \cdot dz}{\sqrt{(x^2 + y^2)(x^2 + z^2)(y^2 + z^2)(x^2 + y^2 + z^2)}}$$

where S_1 is the new singular series, easily deducible from above.

Performing the same substitutions than before, we find:

$$N(x_0, y_0, z_0) = \frac{S_1}{2^4}$$

$$\int_{\sqrt{3}}^{D} \frac{dr}{r^2} \int_0^{\varphi_0} d\varphi \int_0^{\vartheta_0} d\vartheta \frac{1}{\sqrt{(\sin^2\varphi \cdot \cos^2\vartheta + \cos^2\varphi)(\sin^2\varphi \cdot \sin^2\vartheta + \cos^2\varphi)}} = 0(1)$$

for large D.

Hence, it is to be expected only a finite quantity of solutions. In fact, no solutions has been found with smallest side $<2^{32}$, (Internet).