1. ELEMENTS OF POLYGONAL TRIGONOMETRY

1.1. Introduction

More than 60 years ago Professor Valeriu Alaci of "Politehnica" University of Timisoara, Romania developed the "Quadratic Trigonometry".

At that time, in this new chapter of Mathematics, he introduced a "Trigonometry" which is based on a "Trigonometric Square" inscribed in a unit radius circle, as the classical trigonometry is based on a "Trigonometric Circle" of radius one. This "Quadratic Trigonometry" which we denote with (QT), was for the first time presented and published in a book with about 250 pages [1].

Generalizing the idea of the "Trigonometric Square" to the "Trigonometric Polygon", in this chapter we introduce the basic elements of a "Polygonal Trigonometry" which we denote by (PT). At this time we will only introduce the definitions of these new trigonometric functions and establish some basic relationship in this (PT).

1.2. Fundamental relations in (QT)

As we mentioned above, (QT) is based on an inscribed square in a circle of radius one and is positioned in relation to the coordinate axes as they are represented in Figure 1.1.

Similar to the trigonometric function in the "Circular Trigonometry" (CT), in (QT), with reference to the trigonometric square $\overline{ABCD}$, we have: $sq\,\alpha = \frac{MM'}{OA}$ and since $OA = 1$ we have $sq\,\alpha = MM'$. We denoted with $sq\,\alpha$ the function "Quadratic Sine" of angle α.

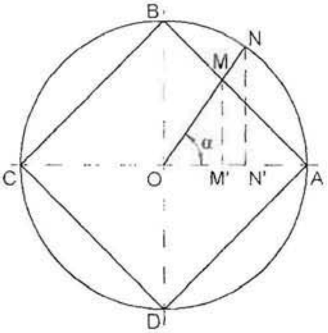

Fig. 1.1. Reference square and its circumscribed circle in the "Quadratic Trigonometry" (QT).

In the same way, for the function "Quadratic Cosine" of angle α we have $cq\,\alpha = OM$. From the triangles $\overline{OMM'}$ and $\overline{ONN'}$ we have a first fundamental relation of (QT), thus:

$$tq\,\alpha = \frac{sq\,\alpha}{cq\,\alpha} = \frac{\sin\alpha}{\cos\alpha} = tg\,\alpha \tag{1.1}$$

With this relation we make the connection between (QT) and (CT).

We denoted (CT) the "Classical Trigonometry" or "Conventional Trigonometry", which is based on the trigonometric circle with center at 0 and radius one. Thus $OA = OB = OC = OD = 1$, which we also represent in Figure 1.1.

Also in (QT) the functions "Quadratic Secant" and "Quadratic Cosecant" are:

$$\sec q\,\alpha = \frac{1}{cq\,\alpha} \text{ and } \operatorname{cosec} q\,\alpha = \frac{1}{sq\alpha}.$$

The second fundamental relation of (QT) results from the right triangles $\overline{OMM'}$ and $\overline{MAM'}$ the last being isosceles with $MM' = AM'$. Since $MM' = sq\,\alpha, OM' = cq\,\alpha$ and $OM' + M'A = 1$ we have:

$$sq\,\alpha + cq\,\alpha = 1 \tag{1.2}$$

It is known that in (CT) the fundamental relations corresponding to (1.2) of (QT) is:

$$\sin^2\alpha + \cos^2\alpha = 1. \tag{1.3}$$

1.3. Polygonal Trigonometry (PT) foundations

(PT) is based on the regular "Trigonometric Polygon" of the number of equal sides n being divisible by four. This trigonometric polygon is inscribed in a circle with radius one. Its position in relation to the reference rectangular axes is represented in Figure 1.2.

Thus the number of sides of the trigonometric polygon is $n = 4 \cdot m$, where m is a positive integer.

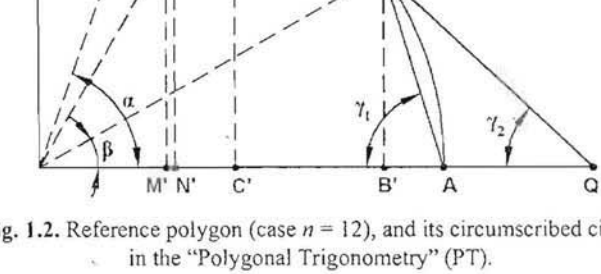

Fig. 1.2. Reference polygon (case $n = 12$), and its circumscribed circle in the "Polygonal Trigonometry" (PT).

For the trigonometric square $n = 4$ and for the trigonometric circle $n = \infty$.

We denote the function "Polygonal Sine" of angle α with $sp(n)\alpha$, and the function "Polygonal Cosine" of angle α with $cp(n)\alpha$, where n represents the number if sides of the trigonometric polygon considered. Thus, using this new notation we can write $sp\,\alpha = sp(4)\alpha$ and $cp\,\alpha = cp(4)\alpha$, respectively for (QT).

Also, we can write $\sin\alpha = sp(\infty)\alpha$ and $\cos\alpha = cp(\infty)\alpha$ respectively for the (CT) case.

Figure 1.2 refers to a trigonometric polygon with twelve sides inscribed in a trigonometric circle with center at 0. To improve the clarity of the figure we represent the first quadrant only $\left(\alpha = 0, \ldots, \frac{\pi}{2}\right)$ of the trigonometric reference.

In this paper we intend to analyze the trigonometric functions variations in the first quadrant only.

1.4. Fundamental relations in (PT)

To obtain relation of general order among the polygonal trigonometric functions we say nothing about the fact that in Figure 1.2 we represented the first quadrant of a trigonometric polygon with twelve sides. In this way instead of $sp(12)\alpha$ and $cp(12)\alpha$ we will use the general notations $sp(n)\alpha$ and $cp(n)\alpha$. We number the circle sectors whose chords are the sides of the considered trigonometric polygon with 1, 2, 3, ..., i where in the trigonometric meaning "i" represents the number of the sectors where the considered angle α opens up. If we refer to the first quadrant we have $i_{\max} = \frac{n}{4}$.

Considering Fig. 1.2 we have

$$sp(n)\alpha = \frac{MM'}{OA} = MM' \text{ and } cp(n)\alpha = \frac{OM'}{OA} = OM'.$$

Also, $tg\,p(n)\alpha = \frac{sp(n)\alpha}{cp(n)\alpha}$.

From the similarity of the triangles $\overline{OMM'}$ and $\overline{ONN'}$ we obtain:

$$\frac{sp(n)\alpha}{cp(n)\alpha} = tg\,\alpha\,. \tag{1.4}$$

Relation (1.4) is similar to (1.1) of (QT).

To establish the second fundamental relation of (PT) we see that:

$$OM' + M'C' + C'B' + B'A = 1\,. \tag{1.5}$$

This relation is valid for the situation represented in Fig. 1.2 (Trigonometric Polygon with $n = 12$).

For the situation regarding trigonometric polygons with a larger number of sides, we have a larger number of line segments of category $\overline{C'B'}$ and $\overline{B'A}$ respectively, which represent the projections on the (OA) axis of some integer sides of the trigonometric polygon.

We mention that we denoted with γ_i the angle $\widehat{MCP}$, but we did not mark this angle γ_i on Figure 1.2 because we wish to have a clear picture in Figure 1.2.

We notice that $\beta = \frac{2\pi}{n}$ and from the triangle $\overline{OCQ}$ we have $\gamma_2 = \pi - 2\beta - \delta$. Likewise the angle $\delta = \frac{\pi - \beta}{2} = \frac{\pi}{2} - \frac{\pi}{n}$. Making all the substitutions we obtain $\gamma_2 = \frac{\pi}{2\pi}(n-6)$.

For the general case we have:

$$\gamma_i = (n - 4i + 2)\frac{\pi}{2n}. \quad (1.6)$$

We remember that $OM' = cp(n)\alpha$, and from the triangle $\overline{MCP}$ we have:

$$M'C' = \frac{sp(n\,\alpha) - \sin\left[(i-1)\frac{2\pi}{n}\right]}{tg\,\gamma_i}. \quad (1.7)$$

From triangle $\overline{CBB''}$ we have:

$$C'B' = \frac{\sin\left[(i-1)\frac{2\pi}{n}\right] - \sin\left[(i-2)\frac{2\pi}{n}\right]}{tg\,\gamma_{i-1}}. \quad (1.8)$$

where

$$\gamma_{i-1} = \left[n - 4(i-1) + 2\right]\frac{2\pi}{n}. \quad (1.9)$$

Applying this reasoning further for sectors with order numbers smaller and smaller compared with i and considering relation (1.6) as having a general character, we can deduce the following relation, of general order, between $sp(n)\alpha$ and $cp(n)\alpha$:

$$cp(n)\alpha + \frac{sp(n)\alpha - \sin\left[(i-1)\frac{2\pi}{n}\right]}{tg\gamma_i} + \sum_{j=1}^{j=i-1} \frac{\sin\left[(i-j)\frac{2\pi}{n}\right] - \sin\left[(i-j-1)\frac{2\pi}{n}\right]}{tg\gamma_{i-j}} = 1. \quad (1.10)$$

where: $i - j \geq 1$.

If in relation (1.10) with the help of (1.9) we perform $n = 4$ (square) we obtain relation (1.2) characteristic of (QT). Consequently, the fundamental relations in (PT) are relations (1.4) and (1.10).

Using these relations we can calculate the values of the trigonometric functions in (PT).

In Figure 1.3 we represented graphically the function $cp(n)\alpha$ (in the first quadrant) for $n = 4$, $n = 8$ and $n = 12$ and for comparison the function $\cos\alpha = cp(\infty)\alpha$.

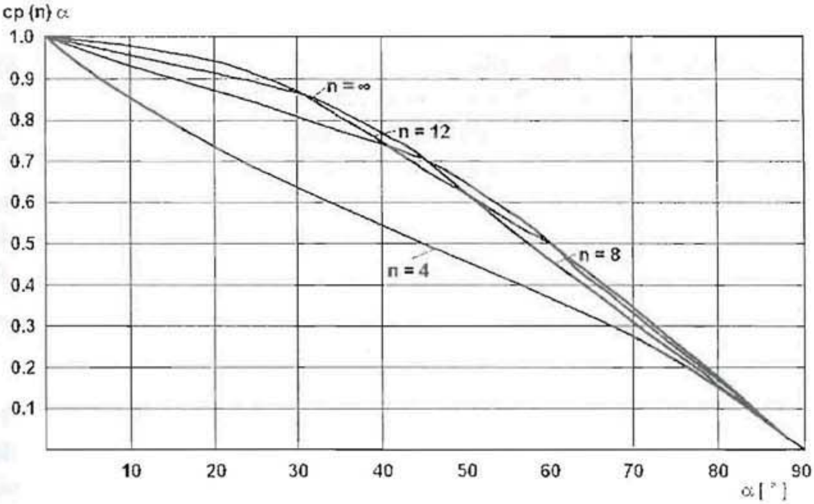

Fig. 1.3. The representation of the functions $cp(n)\alpha$ for $n = 4$, $n = 8$, $n = 12$ and $n = \infty$, in the first quadrant $\left(\alpha = 0,\ldots,\dfrac{\pi}{2}\right)$.

We see that on the points which mark the contact on the circumscribed circle with the trigonometric basic polygon (see Figure 1.2) we have $cp(n)\alpha = \cos\alpha$.

1.5. Conclusions of Chapter 1

"Quadratic Trigonometry" (QT), elaborated by the Romanian Professor Valeriu Alaci is based on the "Trigonometric Square" as well as the "Classical Trigonometry" (CT) is based on the "trigonometric Circle".

In this chapter 1 we elaborated the basic elements of the "Polygonal Trigonometry" (PT), which was developed by using a reference geometrical figure, any regular polygon whose number of sides is a multiple of 4. In this way (QT) become particular case of (PT) when the number of sides of the trigonometric polygon is the minimum possible.