12. PARATRIGONOMETRIC FUNCTIONS RELATIVE TO THE FINITE SPIRALS AS THE BASIC TRIGONOMETRIC FIGURES

12.1. Introduction

In Chapter 6 we analyzed the paratrigonometric functions represented in the Chartesian coordinates and we showed the connection between these functions and The Basic Trigonometric Figures (BTFs). These last ones in their turn were also represented in the Chartesian coordinates.

We also, recall that the fundamental relations in the Paratrigonometry are the following:

$$|spr_k\alpha|^k + |cpr_k\alpha|^k = 1 \tag{12.1}$$

$$tpr_k\,\alpha = tg\,\alpha \tag{12.2}$$

where $spr_k\,\alpha$ is "the paratrigonometric sine of order k of the angle α", $cpr_k\,\alpha$ is "the paratrigonometric cosine of order k of the angle α" and $tpr_k\,\alpha$ is "the paratrigonometric tangent of order k of the angle α". The order k can have values in the domain $0 \le k \le \infty$. Important particular cases are represented when k = 2 (The Classical Trigonometry – CT) and k = 1 (The Quadratic Trigonometry – QT).

The Basic Trigonometric Figures (BTFs) of the corresponding paratrigonometric functions are in their turn expressed in Chartesian coordinates by the relation:

$$|y|^k + |x|^k = 1\,. \tag{12.3}$$

In the CT (k = 2) the corresponding BTF is a circle having its radius R = 1. In the QT (k = 1) BTF is a rhombus with all its angles being right angles, which is inscribed in a circle of the radius R = 1. For any other values of k the BTFs are "rhombuses" with curved sides, which are convex for $1 < k \le \infty$ and concave for $0 \le k < 1$.

All of the BTFs presently studied in regard with the PRT are symmetric with the Chartesian coordinate axis Ox – Oz.

We intend to study further these non-symmetric BTFs corresponding to these axes. We will bring in our analyse those BTFs of spiral form, which develop between the coordinate point (x = 1; y = 0), for $\alpha = 0$ and the point of the coordinates (x = 0; y = 0), for $\alpha = 2 \cdot K \cdot \pi$, K an integer (K = 1, 2, …, etc.).

Evidently, for the best representation of a spiral we are going to use the polar coordinates.

In Chapter 10 we used the polar coordinates to represent some paratrigonometric functions.

In what follows, we will analyze some paratrigonometric functions which are related with these BTFs under the form of finite spirals, that is that the spirals start and end in very well defined points in the coordinates system, as we have shown above.

12.2. Archimedean Spiral, Logarithmic Spiral and Parabolic Spiral having BTFs role in the Paratrigonometry

The classical mathematical expression for the Archimedean Spiral in the polar coordinates is:

$$\rho = c \cdot \alpha \qquad (12.4)$$

where ρ is the polar radius, α is the angle formed by the polar radius with the polar axis Op (see Fig. 12.1) and c is a constant. In other words, the polar radius varies directly proportional with the angle α.

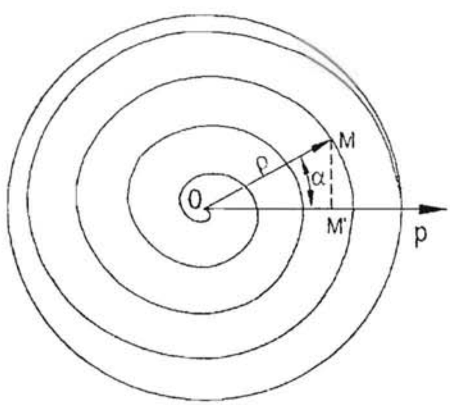

Fig. 12.1. The Archimedean Spiral

We can see that the Archimedean Spiral, expressed in this way – see (12.4), starts from the pole O (for $\alpha = 0$, $\rho = 0$) and tends towards infinity (for $\alpha = \infty$, $\rho = \infty$).

In order to establish the expression for the polar radius which decreases from $\rho = 1$ (for $\alpha = 0$) down to $\rho = 0$ (for $\alpha = 2 \cdot K \cdot \pi$), as we have shown above, we will use for the corresponding spiral (which we name "finite") the relation:

$$\rho = 1 - (\alpha / 2\pi \cdot n) \qquad (12.5)$$

where n represents the number of the complete spires (by 2π rad. each) developed between $\rho = 1$ and $\rho = 0$. In Fig. 12.1 a such spiral is represented for which $n = 4$.

This spiral is developing in a trigonometric direction from $\rho=1$ (for $\alpha=0$) to $\rho=0$ (for $\alpha=8\pi$). For $\alpha>8\pi$ the polar radius ρ becomes negative and this representation does not have any sense.

Another very well known spiral in Mathematics is the Logarithmic Spiral. This one when the angle α increases in a trigonometric sense is represented by the relation:

$$\rho=c\cdot e^{-m\alpha} \tag{12.6}$$

where *c* and *m* are constants greater then 0 (zero), and α is the angle formed by the polar radius ρ with the polar radius Op, as we previously had shown. Since we set the condition that for $\alpha=0$ to have $\rho=1$, from the relation (12.6) results that $c=1$ and the relation (12.6) results that $c=1$ and the relation (12.6) becomes:

$$\rho=e^{-m\alpha}. \tag{12.7}$$

The polar radius ρ tends to 0 (zero) when α tends to $+\infty$. In other way saying, the pole O is the asymptotic pole where the spiral is approaching more and more for α increasing to $+\infty$, but O is never touched by the spiral. This "comes" from $\rho=\infty$ when $\alpha=-\infty$ and passes through the point ($\alpha=0;\rho=1$), towards the pole O, which can be theoretically touched for $\alpha=+\infty$.

Compared with the previous situation (Archimedean Spiral) we accept that in function of the number of the spirals (of 2π rad. each) from which we establish that the entire spiral is formed, the value of ρ is very small and we denote it by $\Delta\rho$. In this case we have:

$$\Delta\rho=e^{-2\pi\cdot n\cdot m}. \tag{12.8}$$

In order to determine *m* we take logarithm in the relation (12.8) and we obtain:

$$m=-(\ln\Delta\rho)/2\pi\cdot n. \tag{12.9}$$

Because in every case $\Delta\rho<1$ then the values of *m* will be positive.

Accepting $n=4$, as in the Archimedean Spiral case, we have:

$$m=-(\ln\Delta\rho)/8\pi. \tag{12.10}$$

Considering for example, $\Delta\rho=0.025$ we obtain $m=0.147$.

A Logarithmic Spiral conform the relation (12.7) and having four spires ($n=4$) and thus $m=0.147$, is represented in the Figure 12.2.

A spiral with a similar form to the Archimedean Spiral is the one represented by the following equation:

$$\rho=a\cdot\alpha^{p}+b \tag{12.11}$$

where *a*, *b* and *p* are constant values.

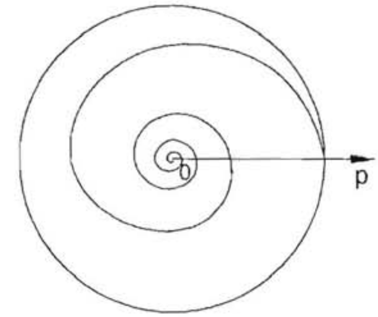

Fig. 12.2. The Logarithmic Spiral.

Because the variable α is raised to a power (p), we namc the corresponding curve to be the "Parabolic Spiral".

If in the relation (12.11) we make $\alpha = 0$, then we get $b = 1$ (in order to have $\rho = 1$). If for $\alpha = 2\pi \cdot n$, we accept $\rho = 0$, then $a = -(1/2\pi \cdot n)^p$. Thus the equation (12.11) becomes:

$$\rho = 1 - (\alpha / 2\pi \cdot n)^p . \qquad (12.12)$$

For $p = 1$ the relation (12.12) is identical with the relation (12.5). If we accept $n = 4$, as above, the relation (12.12) becomes:

$$\rho = 1 - (0.0398 \cdot \alpha)^p . \qquad (12.13)$$

The value for p can be chosen in a such a way that the curve of the function $\rho(\alpha)$ can mathematically model in a very accurate way a specific fenomenon (in Physics, for example) which can be represented by the relation (12.13).

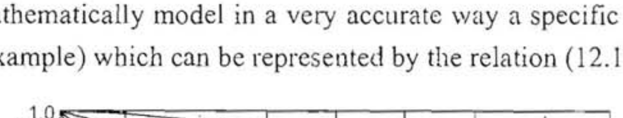

Fig. 12.3. The vector radius as a function of the angle α, $\rho(\alpha)$ for diverse spiral.

In the Figure 12.3 we represent the curves for the function $\rho(\alpha)$ expressed by the above relations for $n = 4$.

Thus:

– the curve a (straight line), relation (12.5) for $n = 4$ and the relation (12.13) for $p = 1$, respectively;

– the curve b, the relation (12.7) for $m = 0.147$ corresponding to $\Delta\rho = 0.025$ – see relation (12.10);

– the curve c, relation (12.13) for $p = 2$;

– the curve d, relation (12.13) for $p = 0.4$.

The value $p = 0.4$ above was chosen by trying, so that the form of the curve d to be the closest to the form of the curve b.

To some Basic Trigonometric Functions (BTF) having spiral forms, analyzed above, exist corresponding specific trigonometric functions, and this will be discussed in the next chapter.

12.3. The Paratrigonometric Spiral Functions

We call the Paratrigonometric Spiral Functions (PSFs) those paratrigonometric functions, which are referred to BTFs with a spiral form. We will analyze those PSFs, which are correlated with the spirals presented in the previous chapter as BTFs.

We denote by $Sps\,\alpha$ the function "Spiral Paratrigonometric Sinus of the angle α", with $Cps\,\alpha$ the function "Spiral Paratrigonometric Cosine of the angle α" and with $Tps\,\alpha$ the function "Spiral Paratrigonometric Tangent of the angle α".

Referring to the Figure 12.1, we see that $Sps\,\alpha$ is equal with the quotient between the magnitude of the line segment MM' and the vector radius ρ. The function $Cps\,\alpha$ is equal with the quotient between the magnitude of the line segment OM' and the vector radius ρ. Between these functions there are the following relations:

$$(Sps\,\alpha)^2 + (Cps\,\alpha)^2 = \rho^2 \tag{12.14}$$

$$Sps\,\alpha \,/\, Cps\,\alpha = Tps\,\alpha = tg\,\alpha\,. \tag{12.15}$$

We see that these relations are similar with the fundamental relations from the Paratrigonometry, which are in relation with the BTFs symmetric to the coordinate axis Ox – Oy [6]. There is also, a similarity of the relation (12.14) with the fundamental relation from the Classical Trigonometry (CT):

$$\sin^2\alpha + \cos^2\alpha = 1\,. \tag{12.16}$$

The distinction between these two relations consists from the fact that, in this case which we are analyzing now, in the right side of the equality instead of a constant (number 1) appears ρ, which is an algebraic function of α.

Also, from Figure 12.1 we observe that the functions $Sps\,\alpha$ and $Cps\,\alpha$ can be expressed by the functions $\sin\alpha$ and $\cos\alpha$ (and ρ) in this way:

$$Sps\,\alpha = \rho\cdot\sin\alpha \tag{12.17}$$

$$Cps\,\alpha = \rho\cdot\cos\alpha\,. \tag{12.18}$$

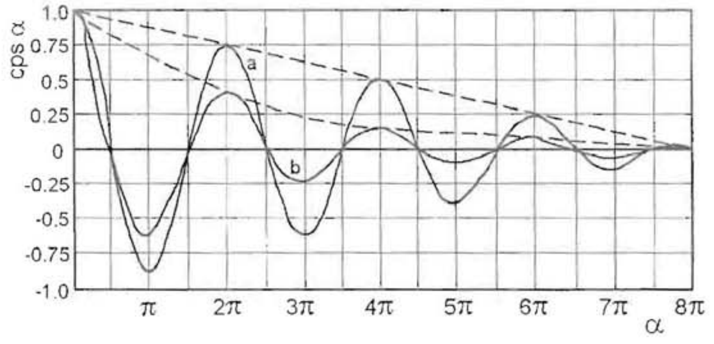

Fig. 12.4. The functions $Cps_A\alpha$ and $Cps_L\alpha$.

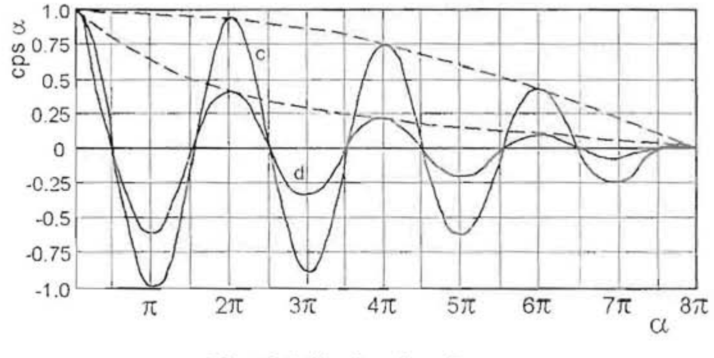

Fig. 12.5. The functions $Cps_P\alpha$.

In the Figures 12.4 and 12.5 the function $Cps\,\alpha$ is represented for the following situations:

• in Figure 12.4:

– a – for the Archimedean Spiral, as a BTF (see Figure 12.1);

– b – for the Logarithmic Spiral, as a BTF (see Figure 12.2);

• in Figure 12.5:

– c – for the Parabolic Spiral, (with $p = 2$) as a BTF;

– d – for the Parabolic Spiral, (with $p = 0.4$) as a BTF.

It is interesting to remark that if we develop the relation (12.18), using for ρ the relation (12.7) which is characteristic to the Logarithmic Spiral, we obtain:

$$Cps_L\,\alpha = e^{-m\cdot\alpha}\cdot\cos\alpha\,. \tag{12.19}$$

We used the notation $Cps_L\,\alpha$ in order to remark the fact that the $Cps\,\alpha$ is to the reference for a Logarithmic Spiral (the index L), as BTF: by analogy, we will also use the notations $Cps_A\,\alpha$ when we refer to an Archimedean Spiral (the index A), as BTS and respectively $Cps_P\,\alpha$ when we refer to a Parabolic Spiral (the index P), as BTF.

The relation (12.19) is exactly the equation for the Amortized Oscillations from Physics and Mechanics respectively if we refer to the Mechanical Systems [23], [24].

It is known that, the mathematical expression which characterizes the Amortized Vibrations is:

$$x = x_0 \cdot \exp(-h \cdot t) \cdot \cos(\omega \cdot t + \varphi) \qquad (12.20)$$

where x is the elongation, x_0 is the variations amplitude, h is the amortization factor, t is the time, ω is the pulsation (circular frequency), φ is the initial phase (diphase). For simplification if we accept $x_0 = 1$ and $\varphi = 0$, we obtain the relation:

$$x = \exp(-h \cdot t) \cdot \cos(\omega \cdot t). \qquad (12.21)$$

This relation is similar with the relation (12.19), if we consider $x = Cps_L\, \alpha$, $h \cdot t = m \cdot \alpha$ and $\omega \cdot t = \alpha$, thus $h / m = \omega$. In another words saying $Cps_L\, \alpha$ can represent an amortized vibration, in the case when the physical characteristics of the vibration (h, t and ω) are adequately in the relation (12.19).

Coming back to the Figures 12.4 and 12.5, we see that in the Figure 12.4 "the enveloping curve" of the curve b represents the graphical expression of the relation (12.7). This curve was traced for positive values only of the function $Cps_L\, \alpha$. It is similar with the curve b of Figure 12.3. In the same way "the enveloping curve" of the curve d of Figure 12.5 represents the graphical expression of the relation (12.13), for $p = 0.4$ and it is similar with the curve d of Figure 12.3.

If we choose adequately the value of p, as we have shown before, the curves b and d of Figure 3 are looking very close alike, even if they refer to the different BTFs namely, the Logarithmic Spiral and Parabolic Spiral, respectively.

12.4. Conclusions of Chapter 12

From what we have shown in the previous chapters, we can take the following important conclusions:

12.4.1. In the Paratrigonometry (Chapter 6) beside the Symmetric Basic Trigonometric Figures (BTFs) we can use the asymmetric BTFs with respect to the coordinate axis Ox – Oy. In this paper we analyzed as BTFs the following finite spirals, developed in the trigonometric sense with the values of the angle α between $\alpha = 0$ up to $\alpha = 2 \cdot K \cdot \pi$ (where K is a positive integer number):

– Archimedean Spiral;

– Logarithmic Spiral;

– The spiral which we named "Parabolic Spiral".

The mathematical modeling of these spirals in this chapter was done in the simplest possible manner such as by representing them in the polar coordinates.

The spirals are developing from the polar radius $\rho = 1$ towards $\rho = 0$, even to $\rho = 0$ or tending to this value (for the Logarithmic Spiral).

12.4.2. The Paratrigonometric Functions corresponding to the BTFs mentioned above (Spirals) denoted by $Sps\,\alpha$, $Cps\,\alpha$, etc., are expressed by the product of the vector radius function which characterize the corresponding spiral $\rho(\alpha)$, and the trigonometric functions of the Classical Trigonometry (CT), $\sin\alpha$, $\cos\alpha$, etc.

12.4.3. The function $Cps_L\,\alpha$ referring to the Logarithmic Spiral (from where we have the index L) as BTF, coincide with the mathematical expression of the elongation in the case of the amortized mechanical vibrations.