14. A SIMPLIFIED ALTERNATIVE TO DEVELOP PERIODIC FUNCTIONS IN TRIGONOMETRIC FOURIER SERIES

14.1. Introduction

We know the fact that the periodic functions having a constant period which can't be expressed by a unique equation can be brought to the situation to be expressed by an unique equation using the Fourier Trigonometric Series [18], [20]. Thus, for example, a periodic succession of the line segments graphically represented in Figure 14.1 (named rectangular function) is mathematically expressed by the following equations:

– for $0 < x < \pi$

$$y = b \tag{14.1}$$

– for $\pi < x < 2\pi$

$$y = -b \tag{14.2}$$

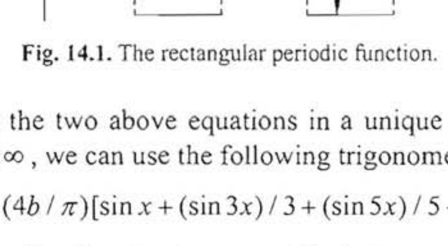

Fig. 14.1. The rectangular periodic function.

In order to put the two above equations in a unique equation valid for the entire domain $0 < x < \infty$, we can use the following trigonometric series:

$$y = (4b / \pi)[\sin x + (\sin 3x) / 3 + (\sin 5x) / 5 + ...]. \tag{14.3}$$

Also, a periodic function having a graphical representation of the form of "saw teeth" (triangular function), as in Figure 14.2, can be expressed by the equation:

$$y = (8b / \pi^2)[\sin x - (\sin 3x) / 9 + (\sin 5x) / 25 - ...]. \tag{14.4}$$

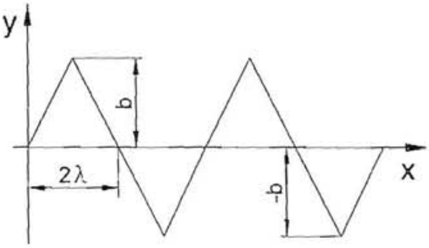

Fig. 14.2. The triangular periodic function.

Practically, any kind of periodic functions can be mathematically expressed by similar equations with the equations (14.3) and (14.4). We know that these equations are enough complicated since for establishing the algebraic terms and factors (non trigonometric) which they comprise is very elaborate. On the other side, these respective equations remain essentially expressed with a certain approximation, their degree of precision depending on the number of the trigonometric terms in the series.

In what follows we will present a simpler method to express mathematically the periodic functions, where we will not use the Fourier series.

14.2. Periodic Functions expressed by using "Matrix Functions", "Matrix Transported Function" and of the "Transport Function"

We consider a function of which the graphical representation in the interval $0 \leq x \leq 2\lambda$ is as given in Figure 14.3 (bold noted curve).

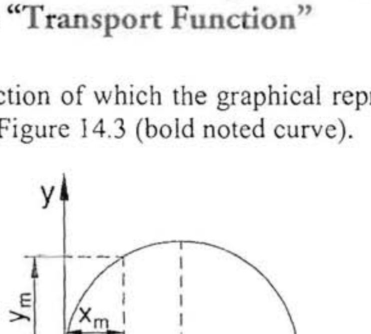

Fig. 14.3. Periodic function formed from the successions of the semi-circles.

We name this "The Matrix Function" (y_M) since this is the one which we intend to transform in a "Periodic Function" (y_P) having the period 4λ. Each semi period of the size 2λ is corresponding to the positive values and respectively to the negative values of the considered function. Thus we have:

$$y_m = y_m(x_m) \tag{14.5}$$

where the variable x_m is in the interval $0 \leq x \leq 2\lambda$.

To transform the function y_m in a periodic function (y_p) having the period 4λ, thus to extend the function y_m by the periodicity [21], we will "transport" the function y_m along the Ox axis in the entire domain up to $x = +\infty$ and we will do this in a such way that alternatively, at equal intervals with the semi period 2λ, the function will change its algebraic sign. In order to do that we use the trigonometric function "sinus" and we will transform the linear measure of the abscissas x into an angular measure. As a consequence, to the semi-period 2λ it will correspond the semi period π. To the variable with linear character x will correspond the variable with angular character υ and the connection between them is the following:

$$\upsilon = (\pi / 2\lambda) \cdot x \tag{14.6}$$

In order to "transport" the matrix function and to change alternatively its sign at the intervals equal with 2λ, as we have shown above, we introduce the following function expressed by the trigonometric function "sinus":

$$y_t = (\sin \upsilon) / |\sin \upsilon| . \tag{14.7}$$

We name the function y_t the "transport function". If we consider relation (14.6) we have:

$$y_t = [\sin(\pi / 2\lambda) \cdot x] / |\sin(\pi / 2\lambda)| . \tag{14.8}$$

The function y_t have all the time the value equal with 1 (one), but its algebraic sign changes at equal intervals with the semi period π, referred to the variable υ, respectively at the intervals equal with 2λ, if we refer to the variable x – see relation (14.8).

We remember that initially the matrix function y_m is expressed on an interval equal with 2λ relatively to the abscissas axis Ox. For this reason we refer the current coordinate to this interval. Thus we can write $x = 2\lambda \cdot n + x_m$, where n is positive integer number belonging to the natural number system, thus $n = 0, 1, 2, \ldots, +\infty$. In this way in order to use the function y_t (x_m), we will arrive to its current value x "moving" this function along Ox axis "steps" equal with 2λ. Considering all these facts it appears "The transported matrix function" which we denote with y_{mt}. In fact, this is the function y_m where instead of x_m we introduce $(x - 2\lambda n)$.

It for example, we have $x = 3 \cdot 3\lambda$, we will write this under the form $x = 2 \cdot \lambda + 1.3 \cdot \lambda$ and we have in this case $n = 1$ and in the matrix function the corresponding abscissa is $x_m = 1.3 \cdot \lambda$.

The desired periodic function which we finally want to have is

$$y_p = y_{mt} \cdot y_t . \tag{14.9}$$

Introducing y_t of relation (14.8) we have

$$y_p = y_{mt} \cdot [\sin(\pi / 2\lambda) \cdot x] / |\sin(\pi / 2\lambda) \cdot x| . \tag{14.10}$$

If we compare the method in the development of periodic functions in Fourier series with the method which we proposed, we observe that in the first case it is done by "composition", in the direction of the Oy axis of a specific periodic function (or of a specific function which is desirable to be periodic), by adding some trigonometric functions, while in our case we proceed in moving in the direction of Ox axis with well established steps, of a know function on an finite interval of the variable x.

In our proposed method case in this paper we eliminate all the disadvantages of the development in Fourier series, mentioned in Subchapter 14.1, in the sense that we have less unnecessary laborious computations and the precision in modeling mathematically this considered function is of 100%.

14.3. Examples of obtained periodic functions starting from Matrix Functions

14.3.1. The case of the line segments parallel with the Ox axis

As a first application example of our method developed in the pervious Subchapter is just the case mentioned in Subchapter 14.1, when we have the Matrix Function $y_{m1} = b$ defined on the interval $0 \le x \le 2\lambda$. As a result of our above discussion, since the function y_{m1} is independent of x we will also have the function y_{mt1} independent of x, thus

$$y_{mt1} = b\,. \tag{14.11}$$

The transport function y_t is given by the relation (14.8) and thus the periodic function y_{p1} will be:

$$y_{p1} = b \cdot [\sin(\pi x / 2\lambda)] / |\sin(\pi x / 2\lambda)|\,. \tag{14.12}$$

Thus for example, for $x = 3.3 \cdot \lambda$ we will have $y_{p1} = -b$, and for $x = 4.1 \cdot \lambda$ we will have $y_{p1} = +b$.

14.3.2. The case when the Matrix Function is represented by a semi-circle

In Figure 14.3 in fact we represented such a function. The equation which represents the semi-circle Matrix Function, valid in the interval $0 \le x \le 2\lambda$ is:

$$(y_{m2})^2 = 2\lambda \cdot x_m - (x_m)^2\,. \tag{14.13}$$

respectively:

$$y_{m2} = (2\lambda \cdot x_m - {x_m}^2)^{1/2}\,. \tag{14.14}$$

Starting with the form (14.14) of the equation and applying the method developed in this paper we obtain the transport Matrix Function y_{mt2} replacing in the relation (14.14) the x_m with $(x-2\lambda\cdot n)$, as we have shown in the previous Subchapter. We remember that n is "the integer part of the quotient in dividing x" by 2λ. The periodic function which represents the semi circle succession of Figure 14.3, using the relation (14.8) will be

$$y_{p2}=y_{mt2}\cdot[\sin(\pi x/2\lambda)/|\sin(\pi x/2\lambda)|]. \qquad (14.15)$$

For example, if we accept $x=2.7\cdot\lambda$ we have $x/2\lambda=1.35$ and thus $n=1$. In consequence, replacing in the relation (14.13) x_m with $x-2n\lambda=2.7\lambda-2\lambda$ $=0.7\lambda$, we obtain $y_{mt2}=0.954\cdot\lambda$. On the other side, the transport function will have the negative value $y_{t2}=-1$ and thus y_{p2} have also a negative value. If we admit $x=5.1\cdot\lambda$, performing the necessary calculations we have y_{mt2} and $y_{t2}=+1$ and thus the periodic function y_{p2} will have a positive value.

14.4. Conclusions of the Chapter 14

In presenting the previous Subchapters we have the following more important conclusions:

14.4.1. The Fourier Method in developing periodic functions in trigonometric series is laborious and does not have a 100 % precision. In order to increase precision there is a need to increase the number of the terms of the respective trigonometric series.

14.4.2. The mathematical modeling Method of a periodic function presented in this Chapter starts from a basic function valid for a limited domain of its variable. We named this function the "Matrix Function" and we proceed to extend its periodicity.

For this reason we introduced two types of functions which were named the "Transported Matrix Function" and respectively, the "Transport Function".

14.4.3. The Method developed in this paper is much simpler then the Fourier Method and ensures a degree of precision in its mathematical modeling of 100%. This fact was illustrated by examples referring to two types of Matrix Functions.