2. REGARDING THE GENERAL CHARACTER OF THE POLYGONAL TRIGONOMETRY

2.1. Introduction

In the chapter 1 the authors presented the basics of the polygonal trigonometry (PT) together with the fundamental relations of PT starting from the foundations of the quadratic trigonometry (QT) elaborated by Professor V. Alaci of the University "Politehnica" of Timisoara, Romania, in the 1930's.

Distinct from the classical trigonometry (CT) which is based on the trigonometric circle with center at O and radius one, QT is developed on a trigonometric square inscribed in a circle with $r = 1$, having its corner at the angles, 0 (zero), $\frac{\Pi}{2}$, Π, $\frac{3\Pi}{2}$, 2Π expressed in radians (respectively 0°, 90°, 180°, 270° and 360° expressed in degrees) of the trigonometric circle.

Similarly, the fundamental relations of PT presented in the chapter 1, were established based on a regular trigonometric polygon inscribed in a circle with $r = 1$. In order to maintain the symmetry conditions this trigonometric polygon must have a number of sides equal to a multiple of four, thus

$$n = 4 \cdot m \tag{2.1}$$

where n is the number of sides of the trigonometric polygon, and m is a positive integer.

Just as for trigonometric square, the corners of the trigonometric polygon after $n/4$, $n/2$ and $3\,n/4$ sides are situated at the angles 0, $\frac{\Pi}{2}$, Π, $\frac{3\Pi}{2}$, of the circle with $r = 1$ circumscribed about the trigonometric polygon.

We make the observation that the trigonometric square is a trigonometric polygon with the minimum possible number of sides ($n = 4$ implies $m = 1$).

Referring to Figure 2.1 where we represent the first quadrant of a trigonometric polygon with 12 sides, $n = 12$ and thus $m = 3$, in the chapter 1 we established the following fundamental relation in PT, valid also for a trigonometric polygon with any number of sides n (respecting $n = 4 \cdot m$):

$$cp(n)\alpha + \frac{sp(n)\alpha - \sin(i-1)\frac{2\Pi}{n}}{tg\gamma_i} + \sum_{j=1}^{j=i-1} \frac{\sin\left[(i-j)\frac{2\Pi}{n} - \sin(i-j-1)\frac{2\Pi}{n}\right]}{tg\gamma_{i-j}} = 1 \tag{2.2}$$

where $cp(n)\alpha$ is the polygonal (p) cosines of α function for the trigonometric polygon with n sides, $sp(n)\alpha$ is the polygonal (p) sines of a function respectively, i is the current number of the circular sector where angle α is situated, counted in the trigonometric direction starting from $\alpha = 0$, and

$$\gamma_i = (n - 4 \cdot i + 2)\frac{\Pi}{2n} \qquad (2.3)$$

$$\gamma_{i-j} = \left[n + 4 \cdot (i - j) + 2\right]. \qquad (2.4)$$

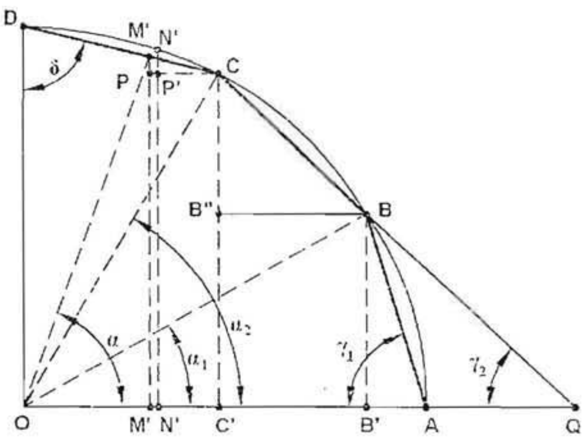

Fig. 2.1. Reference polygon (case $n = 12$) and the circumscribes circle in the Polygonal Trigonometry.

Another important relation valid in all CT, QT and PT is the one which defines the tangent of α function:

$$tg\,\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{sq\,\alpha}{cq\,\alpha} = \frac{sp(n)\alpha}{cp(n)\alpha} \qquad (2.5)$$

where $sq(\alpha)$ and $cq(\alpha)$ are quadratic sines and quadratic cosines of a functions respectively. Thus it follows:

$$tp(n)\alpha = tq(\alpha) = tg\,\alpha\,. \qquad (2.6)$$

In the chapter 1 we showed that applying formula (2.2) in the trigonometric square ($n = 4$) case we obtain the fundamental relation of QT, namely

$$sq\,\alpha + cq\,\alpha = 1\,. \qquad (2.7)$$

In what follows, this chapter will analyze the manner in which the basic elements of PT are applied in the CT case, which in fact represents a particular case of PT (when $n = \infty$), as well as in the QT case which represents the other extremes value of n ($n = 4$).

Evidently, $n_{\min} = 4$ and $n_{\max} = \infty$.

2.2. Classical Trigonometry (CT) a particular limiting case of the Polygonal Trigonometry (PT)

For the above mentioned analysis we start using Figure 2.1 and the geometric elements from this figure, with the help of which we obtain the fundamental relation (2.2) applied in the PT. Thus, in the Figure 2.1 we see that for the triangles $OB'B$ and $OC'C$ having the sharp corners B and C of the trigonometric polygon situated on the trigonometric circle, the Pythagorean Theorem gives the relations:

$$\left(\overline{OB'}\right)^2 + \left(\overline{BB'}\right)^2 = 1 \tag{2.8}$$

$$\left(\overline{OC'}\right)^2 + \left(\overline{C'C}\right)^2 = 1\,. \tag{2.9}$$

In the CT (with reference to the trigonometric circle) we have $\overline{OB'} = \cos\alpha_1$ and $\overline{B'B} = \sin\alpha_1$ and respectively, $\overline{OC'} = \cos\alpha_2$ and $\overline{C'C} = \sin\alpha_2$ and thus:

$$\cos^2\alpha_1 + \sin^2\alpha_1 = 1 \tag{2.10}$$

$$\cos^2\alpha_2 + \sin^2\alpha_2 = 1\,. \tag{2.11}$$

But the trigonometric circle is a trigonometric polygon with an infinite number of sides.

Thus all the points which form this circle could be considered as sharp corners (as well as B and C) of the trigonometric polygon with $n = \infty$. In other words, in the trigonometric circle case, applied to CT, the relations (2.8) and (2.9) and respectively (2.10) and (2.11) are valid for any point on the circle. Therefore, regarding the current angle α, we have:

$$\cos^2\alpha + \sin^2\alpha = 1\,. \tag{2.12}$$

This (2.12) is a fundamental relations of CT.

It follows then that the trigonometric circle represents the upper limit (for $n = \infty$) of the trigonometric polygon. Consequently, CT represents a particular case (for $n = \infty$) of the PT.

2.3. The general character of the Polygonal Trigonometry (PT)

From the previous section of this chapter and from chapter 1 it follows that PT is generally applicable; the mathematical elements which guide us to its fundamental relation make it valid for both QT and CT.

QT is situated at the lower (inferior) limit of the number of sides of the trigonometric polygon ($n = 4$), and CT represents the upper (superior) limit of the PT from this point of view ($n = \infty$).

The fundamental relations of PT have a general character and those of QT and CT represent particular cases of the PT, and they are as follows:

- QT – relation (2.7)
- PT – relation (2.2)
- CT – relation (2.12)

The relations (2.7.) and (2.12) can also be written as follows:

$$\cos^k \alpha + \sin^k \alpha = 1 \qquad (2.13)$$

where $k = 1$ for QT and respectively $k = 2$ for CT.

Considering the above mentioned facts, logically it appears that the formula (2.13) could also be valid in the PT, the exponent value k varying in the closed interval $(1 \le k \le 2)$ and depending on the value of n which characterizes the corresponding trigonometric polygon.

The investigate the validity of such a hypothesis we proceeded to calculated the values of k as a function of the angle α for three distinct values of n from the range $4 < n < \infty$, these values being $n = 8$, $n = 16$ and $n = 24$.

For this reason we use formulas (2.6) and (2.13) and we have:

$$\left[cp(n)\alpha\right]^k = \frac{1}{1+(tg\,\alpha)^k} \qquad (2.14)$$

k is part of the formula (2.14) and as such in order to determinate its values we give k different values in the domain $1 \le k \le 2$. From equation (2.14) using logarithms we obtain the following relation for $cp(\alpha)$:

$$cp(n)\alpha = e^z \qquad (2.15)$$

where

$$z = \frac{\ln R}{k} \qquad (2.16)$$

$$R = \frac{1}{1+(tg\,\alpha)^k}. \qquad (2.17)$$

Value of $cp(n)\alpha$ resulting from formula (2.15) is then compared with the one calculated with the exact relation for this function resulting from formula (2.2) where $sp(n)\alpha$ is replaced with the right side term of the equality

$$sp(n)\alpha = \left[cp(n)\alpha\right] \times tg\,\alpha. \qquad (2.18)$$

With successive trials we obtain an exact value for k. In this way we find that k depends not only on n but also on the value of the angle α.

The variation of k as a function of α, for these three values of n, namely $n = 8$, $n = 16$ and $n = 24$, is graphically represented in Figure 2.2.