3. PERIODIC TRANSTRIGONOMETRIC FUNCTIONS

3.1. Introduction

It is known that many phenomenons in Physics and respectively in technical domains have an oscillation character. In many cases these phenomenons can be mathematically modeled with the help of the trigonometric functions $\sin\alpha$ and $\cos\alpha$ respectively. Examples in this content are the unamortized mechanical vibrations [23] acoustic oscillations, electromagnetic waves etc.

There are some oscillation phenomenons of which mathematical representation does not have a sinusoidal form. In their analysis using the Classical Trigonometry (CT), we apply the decomposition of these functions in Fourier series in order to do the mathematical modeling needed. Let give a single example in this regard, concerning line currents for the electrical transformer with a free current [15]. Intensity variation of such current as a function of the period ωt is represented in Figure 3.1.

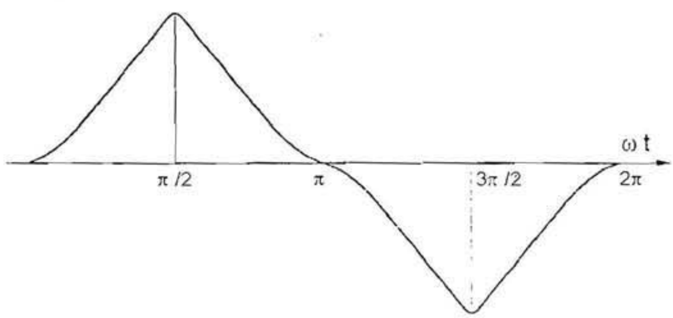

Fig. 3.1. Line current "i" for the electrical transformer with free current, as a function of the temporal period "$\omega\, t$".

On the other side, in Chapter 1 and 2 the authors analyzed the bases of the Polygonal Trigonometry (PT) using the extended characteristic elements of QT [1].

3.2. Two essential relations in the Transtrigonometry

As it is known the basic relations of CT and QT are the following:

– In CT:

$$\sin^2\alpha + \cos^2\alpha = 1 \tag{3.1}$$

– In QT [1]:

$$sq\,\alpha + cq\,\alpha = 1 \tag{3.2}$$

– In PT [2, 4]:

$$sq^k\alpha + cq^k\alpha = 1 \tag{3.3}$$

where k has a variable value included in the domain $1 < k < 2$ [4].

It can be seen that the relations (3.2) and (3.3) are variable for the first trigonometric quadrant ($0 \le \alpha \le \pi/2$). In order that these relations to be valid for all the four quadrants, they must be written under the form:

$$|sq\,\alpha| + |cq\,\alpha| = 1 \tag{3.4}$$

$$|sq\,\alpha|^k + |cq\,\alpha|^k = 1. \tag{3.5}$$

Relation (3.1) can be kept as it is since its availability from the algebraic point of view is preserved also for the negative values of $\sin\alpha$ and $\cos\alpha$ because they are raised to the second power.

On the basis of relations (3.1), (3.2) and (3.3) from above there appears in a logical way the idea to analyze some periodic functions of type $\sin\alpha$, $\cos\alpha$ of CT which should satisfy a similar relation as (3.3) where k would have a constant value (not a variable on as in PT) and which should be included in the domain $1 < k < 2$. At the lower neighborhood of this domain (k = 1) we have QT, and in the upper neighborhood (k = 2) we have CT.

We named Transtrigonometry (TT) the chapter of the Trigonometry which includes the domain between QT (k = 1) and CT (k = 2) and thus it is characterized by k = ct., having values in the domain $1 < k < 2$. The functions of type "sinus α" respectively "cosinus α" we name them "Transtrigonometric sinus α" and "Transtrigonometric cosinus α" and we denote them with $st\,\alpha$ and $ct\,\alpha$. In this way, similarly with the relations (3.4) and (3.5) we will have the relation

$$st\,\alpha^k + ct\,\alpha^k = 1. \tag{3.6}$$

Since k can have any value in the domain $1 < k < 2$ in order to make distinctions between so many situations corresponding the functions $st\,\alpha$, $ct\,\alpha$ etc. by their "order" established by the value of k. The order will be denoted as index to $st\,\alpha$, $ct\,\alpha$ etc. as $st_k\alpha$, $ct_k\alpha$ etc.

Thus, to avoid confusions, relation (3.6) will be written as:

$$st_k\alpha^k + ct_k\alpha^k = 1. \tag{3.7}$$

It can be seen that the relation (3.7) has a general character, and relations (3.4) and (3.5) represent particular cases of that one. Thus, we can write $\sin\alpha = st_2\alpha$, $\cos\alpha = ct_2\alpha$ and $sq\,\alpha = st_1\alpha$, $cq\,\alpha = ct_1\alpha$. In other words, the basic trigonometric functions of QT represent the respective functions "of first order" in TT. The same functions of CT represent the respective functions "of second order" of TT. Thus, the relation (3.7) is the first essential relation of TT.

On the other side, as we have shown in the Chapter 1 and Chapter 2, we can easily prove that the function "tangent" can be included in TT case in the equality:

$$tg\, t_k = tq\,\alpha = tg\,\alpha\,. \tag{3.8}$$

The relation (3.8) represents the second essential relation of TT.

3.3. The characteristics of transtrigonometric functions

For the essential transtrigonometric functions, from (3.7) and (3.8) result the following expressions:

$$st_k\,\alpha = \pm\left(1 + ctg\,\alpha^k\right)^{-1/k} \tag{3.9}$$

$$ct_k\,\alpha = \pm\left(1 + tg\,\alpha^k\right)^{-1/k} \tag{3.10}$$

With relations (3.9) and (3.10) and knowing from CT the values for $tg\,\alpha$ and respectively $ctg\,\alpha$ we can compute the values for the functions $st_k\,\alpha$ and $ct_k\,\alpha$ as functions of angle α, for diverse values of "order" k. The signs + (plus) and – (minus) in the front of formulas (3.9) and (3.10) – right sides – are given in function of the quadrant where angle α is situated. As in the CT, for α situated in I and II quadrants, $st_k\,\alpha$ has positive values ad for α in III and IV quadrants, $st_k\,\alpha$ has negative values. On the other side, the function $ct_k\,\alpha$ has positive values in the quadrants I and IV, and negative values in the quadrants II and III.

In Figure 3.2 the function $st_k\,\alpha$ is represented for values of the angle α (expressed in radians) in the domain $0 \le \alpha \le 2\pi$, and for k = 1 (QT), k = 2 (CT) and k = 1.4 (TT). We see that for k = 2 the function "sinus" is represented by the classical sinusoid and for $1 \le k \le 2$ the sinusoid curves have forms of "Arabian Archivolt" showing "fractures" for $\alpha = \pi/2$ and $\alpha = 3\pi/2$.

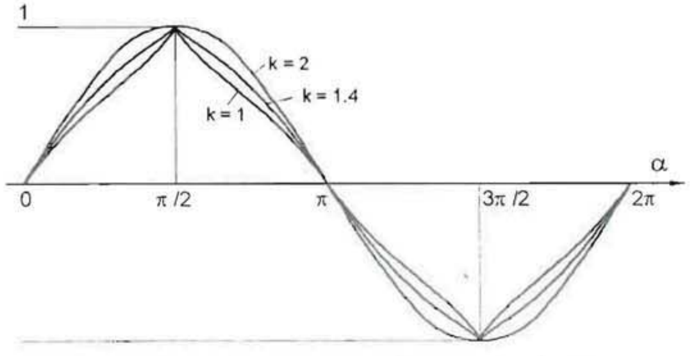

Fig. 3.2. The trigonometric functions $st_k\alpha$ (of order "k") for the values of k = 1, k = 1.4 and k = 2.