4. THE INFRATRIGONOMETRY, AN INFERIOR ORDER NEIGHBORHOOD DOMAIN OF THE TRANSTRIGONOMETRY

4.1. Introduction

It Chapter 3 we established the basic formulas for the relations between the principal trigonometric functions, in a case of a mathematical chapter named Transtrigonometry (TT). These formulas are the following:

$$|st_k\alpha|^k + |ct_k\alpha|^k = 1 \qquad (4.1)$$

$$tgt_k\,\alpha = tgq\,\alpha = tg\,\alpha \qquad (4.2)$$

where $st_k\,\alpha$ is "transtrigonometric sine" (of order k) of an angle α, $ct_k\,\alpha$ is "transtrigonometric cosine" (of order k) of the angle α, $tgt_k\,\alpha = st_k\,\alpha / ct_k\,\alpha$ is "transtrigonometric tangent" (of order k) of an angle α, $tgq\,\alpha = sq\,\alpha / cq\,\alpha$ is "quadratic tangent" of the angle α (see Quadratic Trigonometry QT [1]) and $tg\,\alpha = \sin\alpha/\cos\alpha$ is the tangent of the angle α of the Classical Trigonometry (CT).

Formulas (4.1) and (4.2), above were established on the basic principles analyzed in the Chapter 1 and Chapter 2.

In TT, the value k (named "the order of the trigonometric function [3]) is within the domain $1 < k < 2$. The value $k = 1$ is characteristic to the QT, and $k = 2$ is characteristic to the CT. Next, we will analyze the domain $0 \le k \le 1$ which will generically named Infratrigonometry (IT), being adjacent to TT in the inferior value zone of $k = 1$ (QT).

4.2. The characteristics of the infratrigonometric functions

For the values domain of the order k mentioned above ($0 \le k < 1$) we named the basic trigonometric functions as "infratrigonometric sine of order k at the angle α", denoted $si_k\alpha$ and respectively "infratrigonometric cosine of order k at the angle α", denoted $ci_k\alpha$.

In this case, the formulas (4.1) and (4.2) become:

$$|si_k\alpha|^k + |ci_k\alpha|^k = 1 \qquad (4.3)$$

and

$$tgi_k\,\alpha = tgt_k\,\alpha = tq\,\alpha = tg\,\alpha\,. \tag{4.4}$$

Starting from these formulas, as in [3], we obtain the formulas to calculate the values for $si_k\alpha$ and $ci_k\alpha$ as function of $tg\,\alpha$, and respectively $ctg\,\alpha$ of CT. thus we have:

$$si_k\,\alpha = \pm\left[1/\left(1+|ctg\,\alpha|^k\right)\right]^{1/k} \tag{4.5}$$

and

$$ci_k\,\alpha = \pm\left[1/\left(1+|tg\,\alpha|^k\right)\right]^{1/k} \tag{4.6}$$

The formulas (4.5) and (4.6) applied in IT are similarly with the corresponding formulas of TT (Chapter 3), mentioning that:

– for TT, $1 < k < 2$;

– for IT, $0 \le k < 1$.

Recall that in QT case, $k = 1$, and in CT case, $k = 2$.

In Figure 4.1 we represent graphically "the sine curve" of the function $si_k\alpha$ for $k = 0.4$, and $k = 0.8$ (both in the IT domain) and, for comparison, classical sine curve, $\sin\alpha$. Evidently, in this case, $k = 2$ (CT).

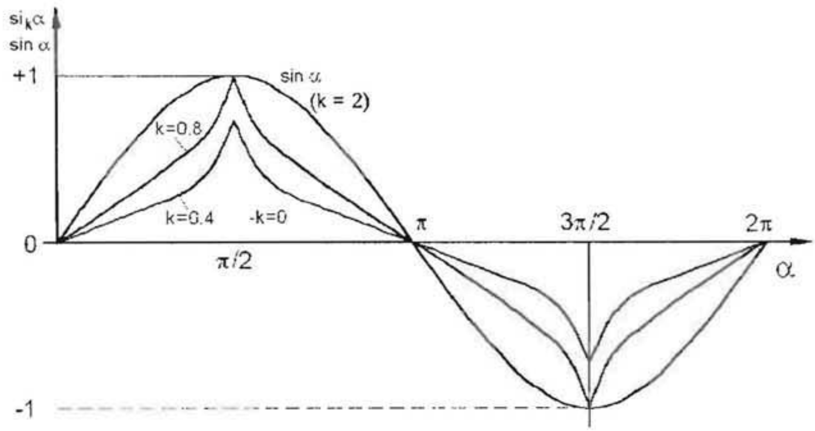

Fig. 4.1. The sine curves for $k = 0$, $k = 0.4$, $k = 0.8$ and $k = 2$.

In Figure 4.1, we also represented the graph of the sine curve for $si_k\alpha$ for $k = 0$ case. This sine curve in fact consists of equal unit length line segments (for absolute value), as we will show detailed in the following chapter. These vertical line segments are periodically situated at the angle α intervals of $\Delta\alpha = \pi$.

We can see that the function $si_k\alpha$ curve has a more pronounced character of an "Arabian Archivolt" than these of the $st_k\alpha$ function (Chapter 3).

The base trigonometric figures represented in the Cartesian coordinate's axis in IT are the same "trigonometric rhombuses with curved sides" as in TT. The distinction consists from the fact that in IT the sides of basis rhombuses are concave in the opposite direction to the reference point O (the coordinate Ox – Oy axis center), while in TT the corresponding sides are concave in the direction to the reference O. remembering that in QT, where $k = 1$, the basis rhombus has the sides (equal sides and equal angles between them) of the line segments form.

Similarly with what we had in Chapter 3 for TT, the basic trigonometric figure in IT is represented by the function:

$$y_k = \pm\left(1 - |x_k|^k\right)^{1/k}. \tag{4.7}$$

In our case (IT), evidently, k has values in the domain $0 \le k < 1$.

In Figure 4.2 we represent the basic "rhombus" with curved sides for $k = 0.4$ and, for comparison, rhombus (with straight sides) for $k = 1$ (QT) and the trigonometric circle for $k = 2$ (CT).

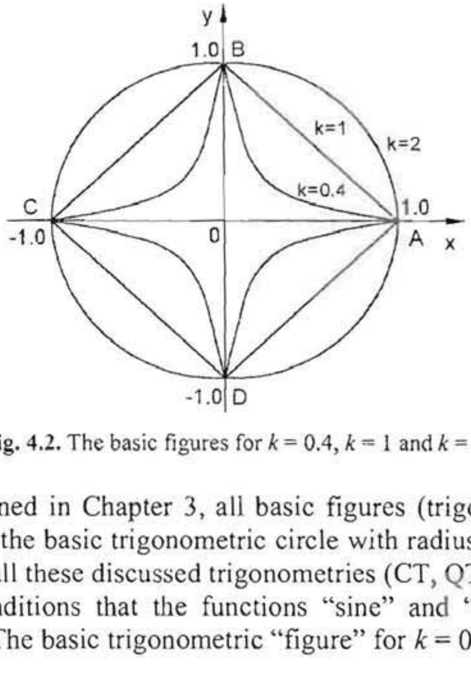

Fig. 4.2. The basic figures for $k = 0.4$, $k = 1$ and $k = 2$.

As we mentioned in Chapter 3, all basic figures (trigonometric rhombuses etc.) are inscribed in the basic trigonometric circle with radius $R = 1$, characteristic for CT. In this way, all these discussed trigonometries (CT, QT, TT, IT) maintained the fundamental conditions that the functions "sine" and "cosine" have values between -1 and $+1$. The basic trigonometric "figure" for $k = 0$ (IT) will be attended in the next chapter.

4.3. The discussion of a special "at limit" case when $k = 0$

The case $k = 0$ represented a limit situation for the values of this parameters in its domain which characterize "the order" of the infratrigonometric functions namely $0 \le k < 1$.

On the other side, k = 1 case (QT) represents the "border" between IT and TT.

Thus, introducing $k = 0$ in formula (4.5) the following situations appear:

4.3.1. When $0 < \alpha < (\pi / 2)$, $ctg\,\alpha$ has a finite value and having $1/k = \infty$ we have $si_0\alpha = 0$.

4.3.2. When $\alpha = 0$, we have the second limit situation (beside the first one determined by k = 0). Thus, having $ctg\,0 = \infty$, the second term of the denominator in formula (4.5) becomes ∞° which represents one of indetermination cases. In order to eliminate this indetermination we apply the method to calculate the "superior" limit and respectively the "inferior" limits [20] or of "right" limit $\left(\lim\limits_{\alpha \to 0^+}\right)$ and respectively "left" limit $\left(\lim\limits_{\alpha \to 0^-}\right)$ as they are named in some professional papers [18].

In our case, as much as for $\alpha \to 0^+$, and for $\alpha \to 0^-$, the values of $\alpha \to 0^-$, the values of $|ctg\,\alpha|$ are finite and we have, as in 4.3.1 above, superior limit $si_0\alpha = 0$ [20]. With this value $si_0\alpha$, for the basic relation (2.1) will give us $ci_0\alpha = 1$.

4.3.3. When $\alpha = \pi / 2$ we have again a limit situation. This time we argue as before, for $ci_0\alpha$ function, thus we will use the basic formula (4.6). We have $ci_0(\pi / 2) = 0$ and using formula (4.3), $si_0(\pi / 2) = 1$.

Conform 4.3.1, 4.3.2 and 4.3.3 from above, in Figure 4.1 we represented function $si_0\alpha$, also. Its graph consists from a succession of horizontal line (overlapped on the abscissa), and the vertical line with the absolute value $|1|$, situated at intervals $\Delta\alpha = \pi$ starting from $\alpha = \pi / 2$. For $ci_k\alpha$, as in CT, the graph is similar with that for $si_k\alpha$, but sliding with $\pi / 2$.

Regarding the basic trigonometric figure of the mathematical model from formula (4.7) we mention that when k = 0, for any value $|x| > 0$ we have $|x|^k = |x|^0 = 1$ and we have $y = 0$. In other words, for any value $|x| > 0$, the basic geometrical figure consists in a line which overlapped on the abscissas axis. For $|x| = 0$ since in formula (4.7) we have $|x| = 0^\circ$, the value of y is indeterminate. Trying to eliminate the indetermination does not bring us to any result and thus, for $|x| = 0$, y can have any value. This means that $|x| = 0$ determine a second line of the basic trigonometric figure which this time overlapped on the ordinates axis.

The same reasoning can be made interchanging x with y in formula (4.7) considering the symmetry of formula (4.3).

We must mention that the basic trigonometrical figure defined above, which is inscribed in the circle with $R = 1$ (of CT) is extended as follows:

– on the abscissa axis $-1 \le x \le +1$

– on the ordinates $-1 \le y \le +1$

In this way we can see that the form is a cross with all four arms equal and each having the unit value (one). In fact, the cross is the essentialized shape of a rhombus.

Regarding Figure 4.2, it is about the line segments OA; OC on the abscissa, and respectively OB; OD on the ordinate.

According to the definition given in the dictionary *a geometric figure is that one in which the curves and surfaces have simple geometric properties*. Under this aspect, the cross is not a proper geometrical figure, but represents the very well known symbol of the Christian Religion.

4.4. Conclusions of Chapter 4

The Infratrigonometry (IT) represents a chapter of the Trigonometry on which there are studied the basic relations established in Transtrigonometry (TT) (Chapter 3), for values of the order k in an adjacent zone to the characteristic zone of TT ($1 < k < 2$) namely for values of k in the domain $0 \le k < 1$.

In this way the quadratic Trigonometry (QT), characterized by $k = 1$, represents the border between IT and TT.

The graphs representing the functions $si_k\,\alpha$ and $ci_k\,\alpha$ have forms which are successions of "Arabian Archivolts" as in TT case. In IT the respective archivolts are sharper than in TT case. The sharp character is much stronger if the value of k is smaller.

At the limit, for $k = 0$, the archivolt is transforming in a succession of horizontal and vertical line segments. Thus for $si_0\alpha$ in $0 \le \alpha < \pi/2$ interval we have $si_0\alpha = 0$. Also, for $\pi/2 < \alpha < \pi$ etc. For α having values $\pi/2$, $3\pi/2$ etc., $si_k\alpha$ have successively $+1; -1$ values etc. (see Figure 4.1).

The basic trigonometric figure in IT, for $0 < k < 1$ is also a rhombus with curved sides as in TT (see Figure 4.2). The distinction between these two situations (IT and TT) is that in IT the concavity of the curved sides is oriented in an opposite sense to the reference point O (the origin of the coordinate axis), while in TT the respective concavity is oriented to the reference O. For the special case, when $k = 0$, the basic trigonometric figure becomes the cross OA – OC – OB – OD represented in Figure 4.2.