5. THE ULTRATRIGONOMETRY, A SUPERIOR ORDER ADJACENT DOMAIN OF THE TRANSTRIGONOMETRY

5.1. Introduction

From Chapter 3 and 4 we know that the basic formulas for TT and respectively IT are the following:

– for TT ($1 < k < 2$)

$$|st_k\alpha|^k + |ct_k\alpha|^k = 1 \tag{5.1}$$

where $st_k\alpha$ is "sine transtrigonometric of order k of the angle α" and $ct_k\alpha$ is "cosine transtrigonometric of order k of the angle α".

– for IT ($0 \le k < 1$)

$$|si_k\alpha|^k + |ci_k\alpha|^k = 1 \tag{5.2}$$

where $si_k\alpha$ is "sine infratrigonometric of order k of the angle α" and $ci_k\alpha$ is "cosine infratrigonometric of order k of the angle α".

Another basis formula, common to all previously described trigonometries (TT, IT, CT and QT) [1] is that one which represents the equality of the "tangent" functions for all above cases namely:

$$tgt_k\,\alpha = tgi_k\,\alpha = tgq\,\alpha = tg\,\alpha \tag{5.3}$$

where tgt is referred to TT, tgi is referred to IT and tgq is referred to Quadratic Trigonometry (QT) [1]; $tg\,\alpha$ is the tangent function of the Classical Trigonometry (CT).

Recall that $k = 1$ is characteristic to QT, and $k = 2$ is characteristic to CT. QT is the border between TT and IT and, as we will see, CT is the border between TT and UT.

5.2. The characteristics of ultratrigonometric functions

Similar to formulas (5.1), (5.2) and (5.3), for UT case (for $1 < k \le \infty$) we have the formulas:

$$|su_k\alpha|^k + |cu_k\alpha|^k = 1 \tag{5.4}$$

$$tgu_k\,\alpha = tg\,\alpha \tag{5.5}$$

where $su_k\alpha$ is "sine ultratrigonometric of order k of the angle α", $cu_k\alpha$ is "cosine ultratrigonometric of order k of the angle α" and $tgu_k\alpha$ is "tangent ultratrigonometric of order k of the angle α". Thus, as in TT and IT cases, starting with formulas (5.4) and (5.5) from above, we obtain the following $su_k\alpha$ and $cu_k\alpha$ in UT:

$$su_k\,\alpha = \pm\left[1/\left(1+\left|ctg\,\alpha\right|^k\right)\right]^{1/k} \tag{5.6}$$

and

$$cu_k\,\alpha = \pm\left[1/\left(1+\left|tg\,\alpha\right|^k\right)\right]^{1/k}. \tag{5.7}$$

The distinction between formulas (5.6) and (5.7) compared with the corresponding values in TT and IT consists only in the domain values of order k.

Similarly with what we had established for TT and IT regarding the basic trigonometric figures, in UT we have valid the following formula:

$$y_k = \pm\left(1-\left|x_k\right|^k\right)^{1/k} \tag{5.8}$$

where k has values in the domain $2<k<\infty$.

Based on formulas (5.4) and (5.5) we construct the graphs for $su_k\alpha$ having $k=3$, $k=8$ and $k=\infty$. For comparison, we give the graph on $\sin\alpha$ function in CT, characterized by $k=2$, as we have shown. These were represented in Figure 5.1.

We mention that for the clarity of figure (the curves for the values of k given above are very close) the "sine" functions were represented for the domain $0<\alpha<\pi/2$ only (first trigonometric quadrant).

If in TT and IT the curves of the functions $st_k\alpha$ and $si_k\alpha$ showed fragments for $\alpha=\pi/2$ and carried on at equal intervals with π, in Figure 5.1 we see that the respective curves are monotonous for $k=3$, $k=8$ and this is generally valid for $2<k<\infty$. The form of curve representing $su_k\alpha$ for $k=\infty$ which in its turn has "segments" and explanations are connected with what we have to say in the next chapter.

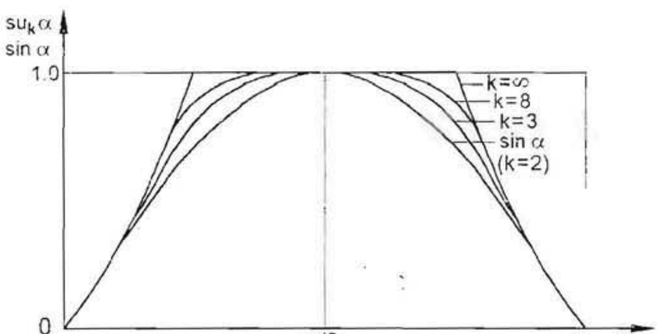

Fig. 5.1. The trigonometric function $su_k\alpha$ for the values of $k=3$, $k=8$ and $k=\infty$ and, for comparison, for $k=2$ (CT).

We can se that the curves for $su_k\alpha$ functions for $2 < k < \infty$ have some prominences (in comparison with the curve of $\sin\alpha$ function) round about the value $\alpha = \pi/2$. These prominences are well-marked when k has a large value. They go to the maximum together with the fragmentation of the respective curve, for $k = \infty$.

The basic trigonometric figures in UT, given by formula (5.8) are represented in Figure 5.2.

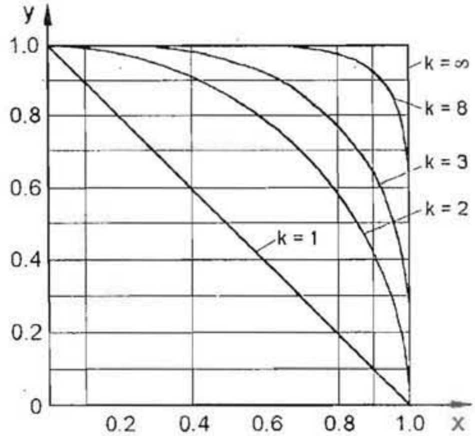

Fig. 5.2. The basic trigonometric figures in UT (for the first trigonometric quadrant) for $k = 3$, $k = 8$ and $k = \infty$ and, for comparison, for $k = 2$ (trigonometric circle of CT).

Again for clarity reasons these figures are represented for the first trigonometric quadrant only. They are referred to the values of the order $k = 3$, $k = 8$ and $k = \infty$. For comparison reason, in the Figure 5.2 we also represent ¼ of the trigonometric circle in CT, characterized by $k = 2$ as well as one of the side of trigonometric rhombus characterized by $k = 1$ in QT.

We see that, as in TT case, the basic trigonometric figures for $2 < k < \infty$ have the concavity oriented towards the reference O (the coordinate axis origin). The case $k = \infty$ will be discussed in the next chapter.

5.3. The discussion of a special "at limit" case when $k = \infty$

In Chapter 4 we discussed a special "at limit" case when $k = 0$. Now, we will discuss the limit case $k = \infty$ which is applied in UT. For this reason, in formula (5.6) we replace $ctg\,\alpha = 1/tg\,\alpha$ and obtain:

$$su_k\,\alpha = \pm\left[|tg\,\alpha| / \left(1 + |tg\,\alpha|^k\right)^{1/k}\right]. \tag{5.9}$$

If we introduce in the denominator of formula (5.9) $k = \infty$ and $\alpha = 0$ we get into an indetermination situation which can not be solved applying L'Hopital rule.

In this case we proceed to calculate the superior limit and respectively inferior limit [20] of $su_k\alpha$ function, more precisely of the denominator in formula (5.9) for $k=\infty$ and $\alpha=0$. In this way we have:

$$\lim_{\alpha\to 0^+}\left(1+\left|tg\,\alpha\right|^{\infty}\right)^0=\left(1+\Delta\right)^0 \tag{5.10}$$

where Δ is very small, but yet $\Delta\neq 0$ and thus $\left(1+\Delta\right)^0=1$.

Similarly, we have

$$\lim_{\alpha\to 0^-}\left(1+\left|tg\,\alpha\right|^{\infty}\right)^0=\left(1+\Delta\right)^0. \tag{5.11}$$

For formula (5.11) we also apply, further, the same reasoning like in formula (5.10). Since both $\lim\limits_{\alpha\to 0^+}\varphi$ and $\lim\limits_{\alpha\to 0^-}\varphi=1$ where $\varphi=\left[1+\left|tg\left(0^{\pm}\right)\right|^{\infty}\right]^0$, we also have, by Chapter 4 and [20], $\lim\limits_{\alpha\to 0}\left(1+\left|tg\,\alpha\right|^{\infty}\right)^0=1$. Since at the numerator $tg\,\alpha=tg\,0=0$, we have $su_\infty 0=0$.

For the situation when the angle α has values in the domain $0<\alpha<\pi/4$, we have $0<tg\,\alpha<1$ and thus $su_\infty\alpha=tg\,\alpha$.

Carrying on, in order analyzing $su_\infty\alpha$ function when $\alpha\geq\pi/4$ we need to return to formula (5.6). In this way we first deal with "at limit" case for $\alpha=\pi/4$ applying the method to compute superior limit and inferior limit for the denominator $\left(1+\left|ctg\,\alpha\right|^{\infty}\right)^0$ of formula (5.6) we have $\lim\limits_{\alpha\to(\pi/4)^+}\left(1+\left|ctg\,\alpha\right|^{\infty}\right)^0=1$ as $\left|ctg\left(\pi/4\right)^+\right|<\infty$. Also $\lim\limits_{\alpha\to(\pi/4)^-}\left(1+\left|ctg\,\alpha\right|^{\infty}\right)^0=1$.

Thus, as we shown above, we have $\lim\limits_{\alpha\to(\pi/4)}\left(1+\left|ctg\,\alpha\right|^{\infty}\right)^0=1$ and therefore $su_\infty\left(\pi/4\right)=1$.

In the domain $\pi/4<\alpha<\pi/2$ we have $0<\left|ctg\,\alpha\right|<1$ and therefore $0<\left|ctg\,\alpha\right|^{\infty}<1$. Consequently, we have $1+\left|ctg\,\alpha\right|^{\infty}\neq 1$ and $su_\infty\alpha=1$ (having $1/k=1/\infty=0$.

For the limit case, when $\alpha=\pi/2$, we proceed as above, applying $\lim\limits_{\alpha\to(\pi/2)^+}$ and $\lim\limits_{\alpha\to(\pi/2)^-}$ to the denominator of formula (5.6) namely $\left(1+\left|ctg\,\alpha\right|^{\infty}\right)^0$. Again we will obtain, and in this case, also $su_{\infty}(\pi/2)=1$. When α has larger values than $\pi/2$, namely in the domain $\pi/2<\alpha<\pi$, then $su_k\alpha$ function will have the form which can be found in Figure 5.1 as it happen with the periodic function "sine" in general. Certainly, also for larger values of α this fact is similar, $su_k\alpha$ function successively having negative values (in the domain $\pi<\alpha<2\pi$) and again positive etc.

The function $cu_k\alpha$ – see formula (5.7) – as we know from CT, it is in fact represented again by a "sinusoid" (in our case, of type $su_k\alpha$) but it is shifted by $\Delta\alpha=\pi/2$.

Regarding the basic trigonometric figures (Figure 5.2) mathematically modeled by formula (5.3), $k=\infty$ situation is considered again as a "limit case" and is treated similarly as the case which we have discussed before. Thus, for $x<1$ we also have $|x|^{\infty}<1$ and therefore $y=1$; this represents the horizontal line (parallel with Ox axis) which includes the segment AB of Figure 5.3.

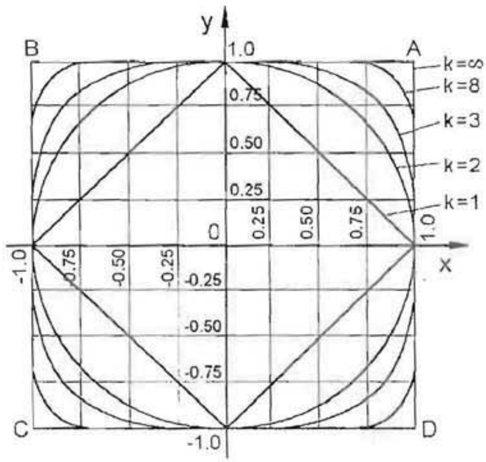

Fig. 5.3. The complete basic trigonometric figures in UT, for $k=3$, $k=8$, $k=\infty$ and, for comparison, for $k=2$ (CT) and $k=1$ (QT).

In this figure, like in Figure 5.2 we represented the basic trigonometric figures completely (in all four trigonometric quadrants) for $k=3$, $k=8$ and $k=\infty$ (UT) and for comparison $k=2$ (CT) and $k=1$ (QT).

"At limit" situation appear for $x=0$. Considering formula (5.8), if we calculate $\lim_{x\to 0^+} y$ and $\lim_{x\to 0^-} y$ we obtain, for both situations, the value 1. This means that the segment AB of Figure 5.3, mentioned above, completes itself with the point of coordinates (0; 1) (see Figure 5.3).

Everything from above are valid for the situation when in front of formula (5.8) the sign + (plus) is taken into consideration. For the situation when we consider the sign – (minus), everything is referred to the line segment CD of Figure 5.3.

If in formula 5.8 we solve for x as a function of y we obtain:

$$x_k = \pm\left(1+\left|y_k{}^k\right|\right)^{1/k}. \tag{5.12}$$

Similarly as above regarding y function where $y=\varphi(x)$ in formula (5.8), we proceed also for the case of function x where $x=\varphi(y)$ in formula (5.12), and obtain the line segments BC and DA. These, together with AB and CD segments form the basic trigonometric figure in UT (the square ABCD) for $k=\infty$.

5.4. Conclusions of Chapter 5

In Ultratrigonometry (UT) we developed the basic relations established in the Transtrigonometry (TT) (see Chapter 3) for the values of the order k comprised in the domain $2<k\leq\infty$ (in TT, $0\leq k<1$).

Thus the classical trigonometry (CT) characterized by $k = 2$, represents the border between TT and UT. The function "sine" of UT denoted by $su_k\,\alpha$, for $k<\infty$, graphically works similar with the function $\sin\alpha$ (of CT), but remarkably outside of $\sin\alpha$ graph, round about the value $\alpha=\pi/2$. This prominence is larger when the value of k is larger (Figure 5.1). In any case, the curves representing function $su_k\,\alpha$ (for $k<\infty$) have a monotonous variation, not having "fragments" as it happen in IT and TT cases (inclusive at the borders between them, in QT). When $k = \infty$, in UT appear "fragment" points of the curve which illustrate $su_k\,\alpha$ function. If we consider the domain $0\leq\alpha\leq\pi$, we find these points when $\alpha=\pi/2$ and $\alpha=3\pi/2$ respectively. For $0\leq\alpha<\pi/2$ the function has the same shape as $tg\,\alpha$. We have the same for $(3\pi/2)<\alpha<2\pi$. In the interval $\pi/2\leq\alpha\leq3\pi/2$ we have $su_k\,\alpha=1$ (in fact $su_\infty\,\alpha=1$).

The basic trigonometric figures in UT are "squares" with curved sides as we can see in Figure 5.2 and Figure 5.3. In $k=\infty$ case, the basic trigonometric figure is a real square ABCD as in Figure 5.3. When the value of k decreases from $k=\infty$ to $k=2$ (CT) the curvature of the square ABCD sides becomes more prominent until it is a circle ($k = 2$). This circle has now the radius $R = 1$ like in CT.