7. REGARDING THE PARATRIGONOMETRIC EQUATION OF THE CIRCLE

7.1. Introduction

It is known that the fundamental relation of Classical Trigonometry (CT), which establish the connection between the values of the functions "sine" (sin) and "cosine" (cos) of the angle α is the following:

$$\sin^2 \alpha + \cos^2 \alpha = 1 . \tag{7.1}$$

If we take in consideration the trigonometric circle (with radius R = 1) and denote axis Ox – Oy, as in Figure 7.1, by similitude with relation (7.1) we have the following well known equation of the trigonometric circle, expressed in an algebraic form:

$$x^2 + y^2 = 1 . \tag{7.2}$$

On the other side, from Paratrigonometry (PRT) – Chapter 6 – we know that symmetric figure (with respect to the line AB of Figure 7.1) of a circle quarter AB (a), with the concavity towards the reference O is the circle quarter AB (b), with the concavity opposite to the reference O. Also, in Chapter 6 we showed that the circle quarter AB (b) can be expressed byt the following algebrical equation in the coordinate system Ox – Oy:

$$x^{0.56} + y^{0.56} = 1 . \tag{7.3}$$

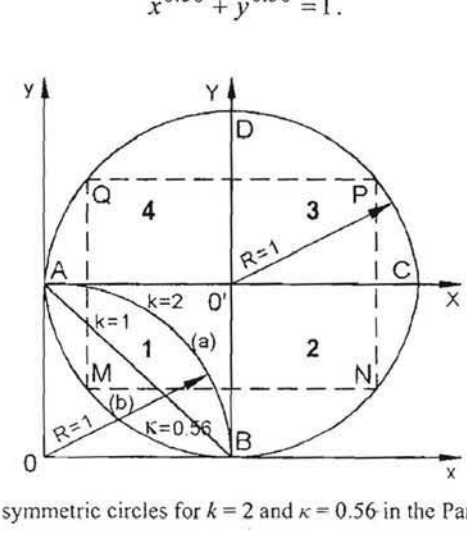

Fig. 7.1. The symmetric circles for $k = 2$ and $\kappa = 0.56$ in the Paratrigonometry.

Recall that the general fundamental equation in Paratrigonometry is:

$$x^{\kappa} + y^{\kappa} = 1 \tag{7.4}$$

for the "order" value κ in the domain $0 \le \kappa < 1$ (Infratrigonometry – IT). The corresponding equation for the order value (denoted with k in this case) in the domain $1 < k \le \infty$ (Extratrigonometry – ET) is:

$$y^{k} + x^{k} = 1 . \tag{7.5}$$

Referring to equations (7.2) and (7.3) and using the established symbols in PRT, we repeat the following relation regarding the symmetry mentioned before:

$$\Sigma\text{im}\,(k = 2) \leftrightarrow \Sigma\text{im}\,\ (\kappa = 0.56) . \tag{7.6}$$

If we refer to Figure 7.1 and take into consideration the coordinate axis system O'X – O'Y we can write the classical equation of a complete circle which contains the arc AB (b), such that:

$$X^2 + Y^2 = 1 . \tag{7.7}$$

We can not do the same thing if we desire to use only one equation of type (7.3) established in PRT and valid only for the quarter of the circle AB (b).

Here below we will establish the paratrigonometric equations of (7.3) type equation for the entire circle with radius $R = 1$.

7.2. Paratrigonometric equations for the entire contour of the circle with radius R = 1

Referring to Figure 7.1, we notice the following:

7.2.1. In the coordinate system Ox – Oy, the quarter of the circle AB (a) is algebraically expressed by equation (7.2) and its "symmetrical" (in the paratrigonometric terms) that is the circle quadrant AB (b) is algebraically expressed by equation (7.3).

7.2.2. If we change the origin of the coordinate axis from O (the coordinate axis being Ox – Oy) to O' (the coordinate axis being O'X – O'Y), the complete circle ABCD is algebraically expressed by equation (7.7).

On the other side, if we return to the coordinate axis Ox – Oy, the equation of the circle ABCD is the following:

$$(x - 1)^2 + (y - 1)^2 = 1 . \tag{7.8}$$

As we have shown in the previous chapter the paratrigonometric equation (7.3) is valid for the circular arc AB (b) only and can not be extended to the entire circle using an equation of type (7.8).

Therefore, to establish the equations of type (7.3) to be also valid and for the rest of three quarters in the circle with center in O', respectively for the circle arcs

BC, CD and DA, we will use the symmetric property of the corresponding circle with respect to O'X and O'Y axis.

Thus, if on the circle arc AB (b) the point M of coordinates x and y satisfies equation (7.3), clearly on the circle arc BC the point N will satisfy the equation (7.3) too if we introduce the coordinates $(2-x)$ and y for this point.

Thus, for the circle arc BC we have valid the equation:

$$(2-x)^{0.56}+y^{0.56}=1. \tag{7.9}$$

For the circle arc CD:

$$(2-x)^{0.56}+(2-y)^{0.56}=1. \tag{7.10}$$

For circle arc DA:

$$x^{0.56}+(2-y)^{0.56}=1. \tag{7.11}$$

The equations (7.3), (7.9), (7.10) and (7.11) can be comprised in a single equation as such:

$$(A+B\cdot y)^{0.56}+(p+q\cdot x)^{0.56}=1. \tag{7.12}$$

If we number all the four quadrants of the circle in Figure 7.1 successively, in a trigonometric sense, starting with the corresponding trigonometric quadrant (TQ) to the arc AB, then we will have the situation given in the table below (Table 7.1)

Table 7.1

The values of coefficients *A*, B, *p* and *q*, depending of *n*

TQ (*n*)	Circle arc	*A*	*B*	*p*	*q*
1	AB	0	1	0	1
2	BC	0	1	2	–1
3	CD	2	–1	2	–1
4	DA	2	–1	0	1

In this table we notice symmetry of the coefficients "p" and "q" with respect of the number "n" which represents the quadrant in reference. If we like to express algebraically this thing, this symmetry directs us to the idea that these respective coefficients can be second degree functions of variable "n". In this case, the graphical representation of these functions, $p(n)$ and respectively $q(n)$, could be that of Figure 7.2.

As it is known, the general equation for such function is, for example, in $p(n)$ case, as follows:

$$p(n)=C_1\cdot n^2+C_2\cdot n+C_3 \tag{7.13}$$

where the coefficients C_1, C_2 and C_3 can be determined applying equation (7.13) for three points of known coordinates "p" and "n", as for example the points "0;1", "2;2" and "2;3".

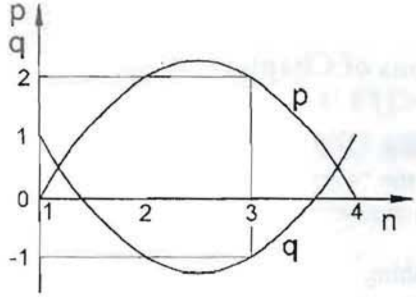

Fig. 7.2. The graphic representation of functions $p(n)$ and $q(n)$.

In this way, from equation (7.13) we obtain three equations with three unknowns C_1, C_2 and C_3. Because of function $p(n)$ symmetry and respectively of curve "p" from Figure 7.2, it was no need to use the "0;4" coordinate point. Solving this three equations mentioned above, we have $C_1 = -1$, $C_2 = 5$, $C_3 = -4$. Thus, equation (7.13) can be written as:

$$p = -n^2 + 5 \cdot n - 4 . \quad (7.14)$$

Similarly, we proceed also for function $q(n)$ and obtain the relation:

$$q = n^2 - 5 \cdot n + 5 . \quad (7.15)$$

For A and B coefficients this above reasoning is not valid and in consequence we applied the following arguments:

For A we can obtain value 0 (zero) in both cases when $n = 1$ and $n = 2$ if we have in view the product $[(n-1) \cdot (n-2)]$. Also with this product we can obtain the correct value for A when $n = 3$. That is $A(3) = 2$. The relation for A becomes applicable also for $n = 4$, if we write it under the form:

$$A = (n-1) \cdot (n-2) / |5 - 2 \cdot n| . \quad (7.16)$$

In our argument to establish the values of the coefficient B we consider the fact that this must change its algebraic sign from + (plus) to – (minus) when it passes from $n = 2$ to $n = 3$. This will happen with a difference of $(5 - 2 \cdot n)$ which also appears in the formula (7.16) above. In order to have values +1 and respectively –1 for the coefficient B we have to divide this difference with its absolute value. Thus we have:

$$B = (5 - 2 \cdot n) / |5 - 2 \cdot n| . \quad (7.17)$$

The circle of radius $R = 1$ in the coordinate system xOy (Figure 7.1) can thus be also represented by equations of degree distinct from two, namely of a degree smaller then 1 ($0.56 < 1$).

In this case we need four distinct equations, one for each circle quadrant individually. This thing was proved above, using specific paratrigonometric mathematical relations.

7.3. Conclusions of Chapter 7

7.3.1. The equation (7.3), fundamental in the Paratrigonometry (PRT) and for which it represents the "symmetry" of the circle represented by equation (7.2), but it is only valid for a quarter of the trigonometric circle of radius $R = 1$.

7.3.2. In establishing a relation of equation (7.3) type, having the exponent value equal with 0.56 for the component terms and with an extended validity in the entire trigonometric circle, we performed a reasoning based on the circle symmetry and thus we found equation (7.12). The coefficients which appear in this equation depend on the reference circle quadrant and can have values 0 (zero); +1; –1 and 2 respectively.

For each of the four circle arcs (corresponding each to one angle at the center equal with $\pi / 2$), which together form a complete circle, we indicate the values of these coefficients in Table 7.1, or they can be determined algebraically as a function of the order number (see Figure 7.1) of the referred quadrant (circle quadrant).