8. SOME DEVELOPMENTS OF FUNDAMENTAL PARATRIGONOMETRIC EQUATION

8.1. Introduction

It Chapter 3 we showed that the fundamental equation which represents the Basic Trigonometric Figure (BTF) in the Transtrigonometry (TT), if we refer only to the first quadrant, is:

$$y^k + x^k = 1 \qquad (8.1)$$

where the value of k is the "order" of the transtrigonometric function of the case analyzed in the respective Chapter. In TT k has values in the domain $1 < k < 2$. From (8.1) results that BTFs in TT are rhombuses with curved sides, with the concavity oriented towards the reference O, which represents the coordinate axis Ox – Oy origin, where the equation (8.1) refers to.

Recall that in the Classical Trigonometry (CT) k =2; BTF in this case is the trigonometric circle having radius $R = 1$. In the Quadratic Trigonometry (QT) $k = 1$ but BTF in this case is the Trigonometric rhombus with straight sides.

For values of k in the domain $0 \le k < 1$ we are in the domain of Infratrigonometry (IT) – see Chapter 4. BTFs in IT are also curved rhombuses as in TT case but the concavity of their sides is oriented in the opposite direction of O.

In the Paratrigonometry (PRT) – see Chapter 6 –, which comprise all of these above mentioned trigonometries, Ultratrigonometry (UT) inclusively having $2 < k \le \infty$ (see Chapter 5) we evidenced some symmetries between BTFs in TT and UT (named together with CT inclusively as Extratrigonometry – ET) and IT. This symmetry is with respect to the straight of the rhombus representing BTF in Quadratic Trigonometry (QT).

In order to distinct BTFs of ET from BTFs of IT we denoted the "orders" of the respective functions by k, in ET case, and by κ , in IT case respectively.

In Figure 8.1 we represented the sides in the first trigonometric quadrant of a BTF from ET – the curve AB(k) – of its symmetry from IT – the curve AB(κ) – and of the trigonometric rhombus – the straight line AB(k = 1) characteristic to QT. Expressing the symmetry of the curves AB(k) and AB(κ) mathematically, we can write $x_1 = 1 - x$ and $y_1 = 1 - y$.

From the formula (8.1) we have

$$y = \left(1 - x^k\right)^{1/k}. \qquad (8.2)$$

If we refer at the curve AB(k), the formula (8.2) become

$$y_1 = \left(1 - x_1^k\right)^{1/k}. \qquad (8.3)$$

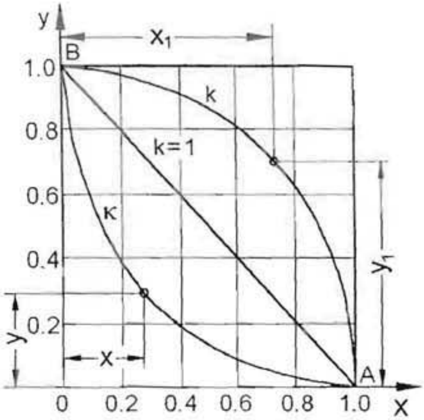

Fig. 8.1. BTFs, in first quadrant, for $k > 1$ (ET), $k = 1$ (QT) and $\kappa < 1$ (IT).

If we refer at the curve AB(κ), the formula (8.2) become

$$y = \left(1 - x^{\kappa}\right)^{1/\kappa}. \tag{8.4}$$

Having in view the above mentioned relations between x_1 and x, and y_1 and y respectively, and the formulas (8.3) and (8.4) we arrive to the following basic relation for the symmetric functions in PRT (see Chapter 6):

$$\left(1 - x^{\kappa}\right)^{1/\kappa} + \left[1 - \left(1 - x^{k}\right)\right]^{1/k} = 1 \tag{8.5}$$

where

$$\kappa = 0.56 - [\ln(k-1)]/6. \tag{8.6}$$

One of the principal reasons of Chapter 6 was to determine formula (8.5).

We have to remember from this chapter that the relation (8.6) and implicity relation (8.5) are valid for the values of k in the domain $1.075 < k \leq 10$.

Evidently, everything what we have discussed before is applied for under unit values of x and y (see Figure 8.1).

Here below we will consider the extension of the validity for equation (8.5) also for over unit values of the variable x.

8.2. Algebraic development of the fundamental equation for BTFs from PRT

If we would like that equation (8.2) to be valid for positive over unit values of x (let denote this case by X) we turn for help introducing a convenient X^{-1} in equation (8.5) and thus we have:

$$\left(1 - X^{-\kappa}\right)^{1/\kappa} + \left[1 - \left(1 - X^{-1}\right)^{k}\right]^{1/k} = 1. \tag{8.7}$$

In this way, we have to use the two equations, namely (8.5), for $0 \le x \le 1$ and (8.7) for $X > 1$.

These two equations can be combined in a single one namely:

$$\left(1-x^{a \cdot \kappa}\right)^{1/\kappa}+\left[1-\left(1-x^{a}\right)^{k}\right]^{1/k}=1 \tag{8.8}$$

where

$$a=(1-x)/\left|1-x\right|. \tag{8.9}$$

We see that for $0 \le x < 1$ result $a = 1$ and thus the equation (8.8) becomes the equation (8.2) and for $x > 1$ we have $a = -1$ and the equation (8.8) becomes the equation (8.7). If $x = 1$ by removing the indetermination in (8.9) we obtain $a = 1$ and we find again the first case when equation (8.8) becomes (8.5).

If we like to express by words relation (8.8) we can say that for any value of k in the domain $1.075 \le k \le 10$ and respecting relation (8.6), the relation (8.8) is valid for any positive rational value of x.

In other words, the equation (8.8) has an infinite number of roots that in all positive rational numbers.

Let denote by $f(x)$ the function of the left side of equality sign in equation (8.8). In this way, the equation (8.8) can be written as:

$$f(x)=1. \tag{8.10}$$

The graphical representation of this function in a coordinate system Ox – Oy, where $y = f(x)$, is the line $y = 1$.

8.3. The validity extension of the BTFs fundamental equation in PRT

The idea to extend the validity of equation (8.2) for values of x greater than the unit by introducing the coefficient "a" given by the relation (8.9) thus obtaining the equation (8.8), can be also useful in formula (8.1).

Thus, this becomes:

$$y^{k}+x^{a \cdot k}=1 \tag{8.11}$$

From this relation we obtain:

$$y=\left(1-x^{a \cdot k}\right)^{1/k}. \tag{8.12}$$

According to what we shown here above, formula (8.12) remain the same in the Extratrigonometry (ET) case, when $1 \le k \le \infty$. In the Infratrigonometry (IT) case formula (8.12) becomes:

$$y=\left(1-x^{a \cdot \kappa}\right)^{1/\kappa} \tag{8.13}$$

where κ is given as a function of k by formula (8.6). By the terms established in the PRT the curve representing the function y given by the formula (8.13), is the "symmetry" of the curve representing the function y given by formula (8.12), or using the symbol of Chapter 6, we can write:

$$\Sigma\text{im}\,(k) \leftrightarrow \Sigma\text{im}\,(\kappa) \tag{8.14}$$

Based on formula (8.14) and using the known result from PRT, evidently we can say that also the converse of what we have said before is valid, that is that the curve associated to the formula (8.12) is the "symmetry" of the curve associated to the formula (8.13).

Recall that this symmetry is formed with the respect of the line AB (k = 1) of Figure 8.1.

If we represent graphically function y of relation (8.12) for $x > 1$, we obtain the curves represented in Figure 8.2.

The curves representing function $y(x)$ for various values of κ and k respectively.

a. $\kappa = 0.236$ **e.** $k = 1.4$
b. $\kappa = 0.56$ **f.** $k = 2.0$
c. $\kappa = 0.713$ **g.** $k = 8.0$
d. $k = 1.0$

In order to have a complete image of this function graphical representation Figure 8.2 we also represented the respective curves for $0 \le k \le 1$, accepting k and κ as parameters with the limits $1.075 \le k \le 10$, as we had before.

In Figure 8.2 we limit our self to x = 4, because of space limit. In any case, for all values of k and κ, the curves for $k > 1$ are asymptotic with respect to the line y = 1. This also results from the relation (8.12) which for $x > 1$, thus $a < 0$ from relation (8.9) at the limit, when $x \to \infty$ it becomes:

$$\lim_{x \to \infty} y = 1. \tag{8.15}$$

We mention that in Figure 8.2 the represented curves are connected by the following symmetric relations: Σim (k = 8) $\leftrightarrow$ Σim (κ = 0.713); Σim (k = 2) $\leftrightarrow$ Σim (κ = 0.56) and Σim (k = 1.4) $\leftrightarrow$ Σim (κ = 0.236).

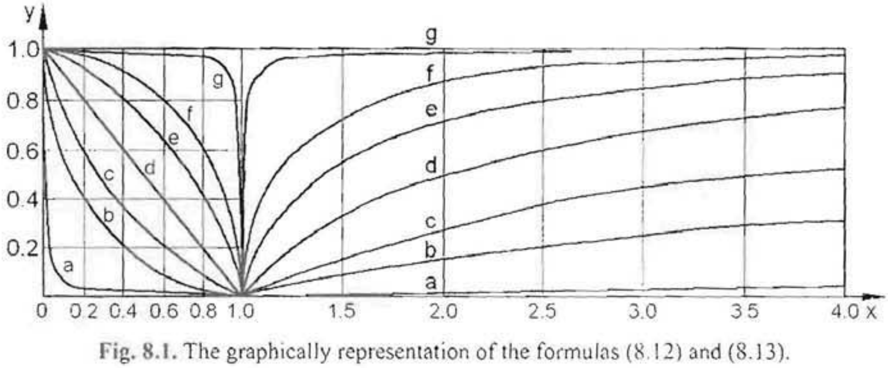

Fig. 8.1. The graphically representation of the formulas (8.12) and (8.13).

It is good to notice the fact that the curves of the function y go faster to the limit $y = 1$, when the values of k and κ respectively are large. Thus, we can see that for $k = 8$, the curve $y(x)$ is very close with the asymptote $y = 1$ starting with $x = 0.5$ and for $k = 2$ this closeness becomes more evident for $x = 4$.

On the other side, in the domain $\kappa < 1$ (thus in IT), for a very small value of κ (for example $\kappa = 0.236$) the curve $y(x)$ will move away very slowly (almost unnoticed for $x < 4$) from Ox axis.

Also, it is good to mention that all curves for $y(x)$ function in the domain characterized by $x > 1$ have their concavity oriented toward Ox axis, while the curves of this function in the domain $0 \le x \le 1$ have the concavity towards the Ox axis respective to the origin O, for $x > 1$ case and in a reverse direction, for $\kappa < 1$ case.

Since, for $x > 1$ the function $y(x)$ curves are developing toward infinity, in one way we can say that they represent the "projection toward infinity" of the curves for the respective function for $0 \le x \le 1$.

8.4. Conclusions of Chapter 8

8.4.1. The basic relation of the Paratrigonometry (PRT) – see Chapter 6 – elaborated on the bases of the principle of the Basic Trigonometric Figures (BTFs) symmetry, represents, in fact, an equation with an infinite number of solutions. In another way said, it is valid for any positive rational value of the variable x.

This means that the corresponding relation, enough complex do, having x as variable, is equivalent with a simple equality: $y = 1$.

8.4.2. In PRT the functions, which form the algebraic expression of BTFs were having until now the values in the domain $0 \le x \le 1$ and $0 \le y \le 1$. Using a convenient calculation these functions can become valid also for the domain $1 \le x \le \infty$ and $0 \le y \le 1$. The representing curves for this functions when $x \ge 1$ start from the values $y = 0$ (for $x = 1$) and progress asymptotically towards $y = 1$ (for $x \to \infty$).