9. THE APPLICATION OF THE PARATRIGONOMETRIC FUNDAMENTAL EQUATIONS IN THE FOUR TRIGONOMETRIC QUADRANTS

9.1. Introduction

It is known from the Chapter 3 and 6 that the fundamental Basic Trigonometric Figures (BTFs) of the Paratrigonometry (PRT), written in the simplest form is:

$$y^k + x^k = 1 \tag{9.1}$$

where y and x are BTF coordinates and k is the "order" of the paratrigonometric function. The equation (9.1) is valid only in the first trigonometric quadrant (TQ), region in which both x and y have positive values.

Recall, as particular cases, the values $k = 2$ (Classical Trigonometry – CT) and $k = 1$ (Quadratic Trigonometry – QT). We see that k = 2 (CT) case the equation (9.1) is valid for all the four TQ, since the even power applied to the variables x and y cancel their negative values. Thus, the something is valid for all even number values of k, that is $k = 2 \cdot n$, when n is a positive natural number.

In Figure 9.1 we represent BTFs for $k = 1$ (in QT), $k = 2.5$ and $\kappa = 0.4924$ in the coordinate system xOy and we use the Roman notation (I – IV) for the four TQs. The value $\kappa = 0.4924$ represents the "symmetry" for value $k = 2.5$ in the signification of the characteristic symmetries in the PRT [6].

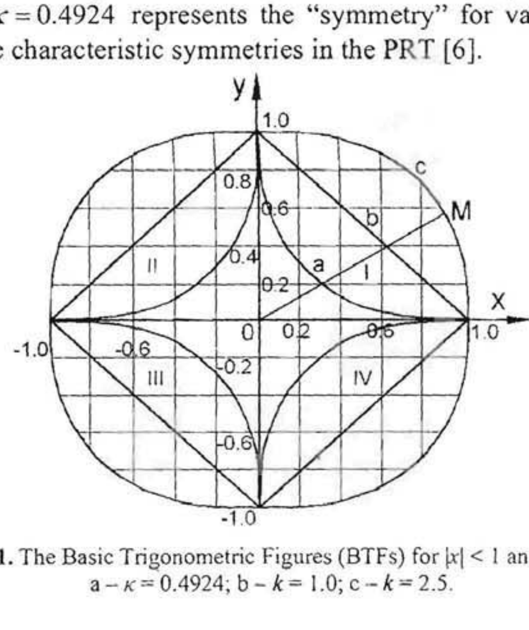

Fig. 9.1. The Basic Trigonometric Figures (BTFs) for $|x| < 1$ and $|y| < 1$: a – $\kappa = 0.4924$; b – $k = 1.0$; c – $k = 2.5$.

We mention that in PRT (see Chapter 6), in order that equation (9.1) to be valid in all the four TQs, was written under the form:

$$|y|^k + |x|^k = 1. \tag{9.2}$$

The equation (9.2) is satisfied for both positive and negative values of the variable x and y.

The problem of the algebraic signs + (plus) and – (minus) is coming when in (9.2) we solve for y in function of x. Thus we have

$$y = \pm\left(1 - |x|^k\right)^{1/k}. \tag{9.3}$$

In this case we must "choose" the sign + (plus) when we refer to TQs I and II and respectively the sign – (minus) when we consider the TQs III and IV. This choice is not mathematically objective, and when we operate with the relation (9.3) we have to use either the real image (geometrical) or the thought image of a BTF.

9.2. The algebraic signs coefficients + (plus) or – (minus)

As we known, the fundamental algebraic equation of the BTFs, written under the form (9.1) or (9.2) has a trigonometric correspondent, the fundamental trigonometric equations.

This establishes the relation between the basic function of type "sinus" and "cosinus" respectively. It is similar with equation (9.1) and (9.2) respectively, the variable y and x being replaced with the functions "sinus" and "cosinus" of the angle α, represented in Figure 9.1.

In PRT these functions are denoted by $spr\,\alpha$ and $cpr\,\alpha$ respectively (see Chapter 6). If in Figure 9.1 we refer to the corresponding BTF for $k = 2.5$, we will have $MN = spr\,\alpha$ and $ON = cpr\,\alpha$.

As we have shown, the equivalent equation to (9.1) is:

$$spr_k^{\,k}\,\alpha + cpr_k^{\,k}\,\alpha = 1 \tag{9.4}$$

where $spr_k\,\alpha$ is the "paratrigonometric sine of order k for angle α" and $cpr_k\,\alpha$ is the "paratrigonometric cosine of order k for angle α".

The equivalent equation with (9.2) is the following:

$$|spr_k\,\alpha|^k + |cpr_k\,\alpha|^k = 1. \tag{9.5}$$

If we solve for $spr_k\,\alpha$ in equation (9.5) we obtain the following relation equivalent to (9.3):

$$spr_k\,\alpha = \pm\left(1 - |cpr_k\,\alpha|^k\right)^{1/k}. \tag{9.6}$$

A similar relation we obtain when we solve for $cpr_k\,\alpha$ in (9.5).

Regarding to the choice of sign + (plus) or – (minus) in (9.6), it is valid in the same discussions performed regarding relation (9.3).

In the relation (9.6) case we can avoid the subjectivity referred when we analyzed relation (9.3) by introducing the following "algebraic sign coefficient":

$$b = (\pi - \alpha) / |\pi - \alpha| \tag{9.7}$$

where the angle α is expressed in radians, thus α [rad]. If angle α is expressed in degrees, the relation (9.7) becomes:

$$b = (180 - \alpha) / |180 - \alpha| . \tag{9.8}$$

Thus relation (9.6) becomes:

$$spr_k\,\alpha = b \cdot \left(1 - |cpr_k\,\alpha|^k\right)^{1/k} . \tag{9.9}$$

By the coefficient b the sign + (plus) or – (minus) in formula (9.6) automatically appears in formula (9.9). That is, if $\alpha < \pi$ (TQs I and II) in relation (9.7) then we have $b = +1$, and when $\pi < \alpha < 2 \cdot \pi$ (TQs III and IV) we have $b = -1$. When we consider relation (9.8), the same reasoning is valid, considering 180° instead of π (in radians) and expressing α in degrees, as we have shown before.

If we consider relations (9.7) and (9.8) for case $\alpha = \pi$, respectively $\alpha = 180^\circ$, the we have an indeterminate situation. In order to eliminate this indetermination we use the method to calculate the limits "superior" and "inferior" respectively, explained in Chapter 4. In this case we have:

$$\lim_{\alpha \to \pi^-} b = +1 \tag{9.10}$$

$$\lim_{\alpha \to \pi^+} b = -1 . \tag{9.11}$$

We see that even in the place $\alpha = \pi$ the algebraic sign changes from + (plus) to – (minus) in formulas (9.7) and (9.8). In $\alpha = 2 \cdot \pi$ case, the same reasoning is valid for the inferior limit. Thus:

$$\lim_{\alpha \to 2\cdot\pi^-} b = -1 . \tag{9.12}$$

For the situation $\alpha > 2 \cdot \pi$ we have to apply to the "periodic" character of the trigonometric and paratrigonometric functions. In this sense, we reconsider the cycle valid for the interval $0 < \alpha < 2 \cdot \pi$. Thus, we can write:

$$\lim_{\alpha \to 2\cdot\pi^+} b = +1 . \tag{9.13}$$

In this way, the angle α varying in a trigonometric sense, when it passes through the value $\alpha = 2 \cdot \pi$, the b coefficient passes from the value $b = -1$ to $b = +1$.

We mention that, for the algebraic sign coefficient above mentioned, we used the letter "*b*" since the letter "*a*" was used in Chapter 8 before, for another algebraic sign.

The problem of the algebraic sign for the function $y(x)$ expressed by the relation (9.3) which represents the mathematical expression of the BTFs can not be resolved in the same way as above. That is because in the corresponding relation the angle α does not interfere as it happens in the trigonometric functions case – se formula (9.6) – in order for the algebraic signs + or – to automatically appear in formula (9.3) in function of trigonometric quadrant (TQ) to which we refer, its is evident that we have to find a coefficient (symbolized by "c") that depends by the order number "n" of the reference TQ.

We express c by a similar relation with relations (9.8) and (9.9) respectively. Thus:

$$c = (m - n) / |m - n|. \qquad (9.14)$$

For c to have the value $c = +1$ in $n = 1$ and $n = 2$ cases (TQs I and II) we must have $m > 2$. Similarly, for $c = -1$ in $n = 3$ and $n = 4$ cases (TQs III and IV) we must have $m < 3$. Both above conditions are thus satisfied when $2 < m < 3$. If $m = 2.5$, in this case formula (9.14) becomes:

$$c = (2.5 - n) / |2.5 - n| \qquad (9.15)$$

and the relation (9.3) becomes:

$$y = c \cdot \left(1 - |x|^k\right)^{1/k} \qquad (9.16)$$

where c is given by (9.15), n being the order number of TQ, from 1 (TQ I) to 4 (TQ IV).

With a similar problem to establish some coefficients containing also the algebraic signs + or –, we have meet in Chapter 7. Summing up, of everything what we have mentioned above regarding the algebraic sign coefficients introduced in Paratrigonometry (PRT) in this Chapter and Chapter 8 we remember the following:

The coefficient a is given by the formula

$$a = (1 - x) / |1 - x|. \qquad (9.17)$$

It was introduced in Chapter 8 in order to have only one relation linking the functions representing the BTFs and their "symmetries", as well for unit values of x, as for their upper-unit values.

The coefficient b was given by formulas (9.7) and (9.8) above and is used to establish "automatic" mathematically, the algebraic sign + or – of the $spr_k\,\alpha$ as a function of the values of the angle α – see relation (9.9).

The coefficient c given by relation (9.15) above is used to establish the algebraic sign of the function y which represents BTF corresponding to the paratrigonometric functions – see relation (9.16).

Figure 9.2 is an application example of the fundamental equation for BTFs, in the general case $-\infty < x < \infty$ (see Chapter 8), using the algebraic signs coefficients established in this Chapter.

In this Figure 9.2, we represented in all four TQ, the BTFs for $k = 2$, $k = 1$ and $\kappa = 0.56$. This last one ($\kappa = 0.56$) represents the "symmetry", in the paratrigonometric meaning, of a BTF characterized by $k = 2$. In Chapter 8 there is a similar figure, but only regarding to TQ I.

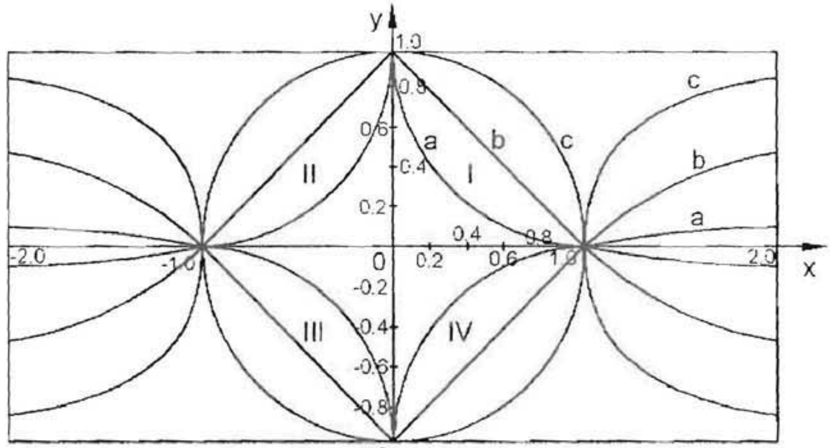

Fig. 9.2. BTFs for $0 < x < \infty$ and $|y| < 1$:
a – $\kappa = 0.56$; b – $k = 1.0$; c– $k = 2.0$.

9.3. Conclusions of Chapter 9

Currently, we attribute to the trigonometric functions the algebraic signs + (plus) or – (minus) in an intuitive way according to the Trigonometric Quadrant (TQ) – I to IV – which we are considering. Thus, as we well know, the function $\sin\alpha$ has positive values in the TQs I and II and negative values in the TQs III and IV. The function $\cos\alpha$ has positive values in the TQs I and IV and respectively negative values in the TQs II and III. The other trigonometric functions, being expressed in function of $\sin\alpha$ and $\cos\alpha$, will have the resulting algebraic signs.

Also, the algebraic signs of the coordinates x and y of the points on the curves which form the Basic Trigonometric Figures (BTFs), that is trigonometric circle in the Classical Trigonometry (CT) case are attributed in function of the reference TQ. Thus, the abscissa x has positive values in the TQs I and IV and negative values in the TQs II and III. The ordinate y has positive values in the TQs I and II and negative values in the TQs III and IV.

For all the above cases and with validity extended to the entire domain of the Paratrigonometry (PRT) in order to result mathematically (not intuitively) these mentioned above algebraic signs, we established the algebraic signs coefficients "b" and "c". The coefficient b is applied in the paratrigonometric functions case $spr\,\alpha$ and $cpr\,\alpha$, respectively. It is expressed as a function of angle α. The coefficient c is applied in the coordinates x and y case which define the referred BTF. It is expressed in function of the order number "n" of TQ in discussion. Evidently, for TQ I we have $n = 1$ and so on.

In this way, all the mathematical relations derived in the previous Chapters 3 and 8, which for simplification were established for TQ I, can be applied in all the others TQs (II – IV), where the algebraic signs + (plus) or – (minus) can be established mathematically, by computing the coefficients b and c.