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Putting last digits first yields multiple

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To Professor Mauro Picone on his 90th anniversary

RIASSUNTO - *Si considerino i numeri naturali* $A = a_{n-1} \dots a_1 a_0$, *espressi in base g. Oltre alla base* $g \geq 2$ *siano assegnati un numero naturale* $n_0 \geq 1$, *un numero* A_0 *di* n_0 *cifre ed un moltiplicatore positivo m. Il problema è quello di trovare quei numeri A rappresentati da più di* n_0 *cifre, che terminino con l'insieme di cifre* A_0, *per i quali il multiplo secondo il moltiplicatore m si ottiene traspartando al primo posto l'insieme di cifre* A_0. *Si dimostra che condizione necessaria e sufficiente per l'esistenza di numeri A di tale tipo è che risulti* $m \leq a_{n_0-1}$ *(essendo* a_{n_0-1} *la prima cifra di* A_0*). Sotto tale ipotesi viene data la costruzione di tutti i numeri A di tale tipo.*

Introduction.

Starting-point for the quite amusing little problem treated here was a paper by LEON BERNSTEIN [1] about what he called *multiplicative twins*, i. e., numbers like

$$A = 142857 \text{ with } 5A = 714285$$

or

$$A = 102564 \text{ with } 4A = 410256,$$

for which putting the last digit first yields a multiple. He gave a systematic determination of all such numbers and developed an algorithm for finding their digits.

Recently the first author of the present paper generalized Bernstein's results to sets of several last digits instead of a single one. When she submitted her paper to Crelles Journal with a dedication to its editor, the second author, on the occasion of his 75 th birthday, he found that the main results and their proofs remained valid for any

basis $g \geq 2$ instead of only the common basis $g=10$ of our number system. So he proposed a joint publication instead of that dedication.

§ 1. The general problem.

Consider positive rational integers

$$A = \alpha_{n-1} \dots \alpha_1 \alpha_0 \quad (0 \leq \alpha_\nu < g \text{ for } \nu = 0, 1, \dots, n-1),$$

expressed in the g-adic digit system; for brevity they will be simply called *numbers*. Besides of the *basis* $g \geq 2$, let be given a *digit-number* $n_0 \geq 1$, a number A_0 of at most n_0 digits, and a positive rational integer $m \geq 1$ as *multiplicator*.

The problem is to find those numbers A of more than n_0 digits with last digit-set A_0 (brough up to exactly n_0 digits by digits 0 in front) whose m-fold is obtained by putting the digit-set A_0 first:

$$\text{(1a)} \quad A = A_1 g^{n_0} + A_0 \quad \text{with} \quad A_0 < g^{n_0} \quad \text{and} \quad g^{n_1-1} \leq A_1 < g^{n_1}$$

(A_1 has exactly n_1 digits),

and the

$$\text{(1b)} \quad mA = A_0 g^{n_1} + A_1 .$$

§ 2. Necessary condition.

THEOREM 1. *Necessary for the existence of numbers A satisfying* (1a), (1b) *is the inequality*

$$\text{(2)} \qquad m \leq \alpha_{n_0-1} \ \textit{(first digit of } A_0\textit{), hence certainly } m < g.$$

PROOF. According to (1a), the request (1b) means the equation

$$\text{(3)} \qquad A_0 g^{n_1} + A_1 = m(A_1 g^{n_0} + A_0).$$

By the inequalities for A_1 in (1a), this equation yields on the one hand the inequality $A_0 g^{n_1} + A_1 > m g^{n_1+n_0-1}$ and on the other hand the inequality $A_0 g^{n_1} + A_1 < (A_0 + 1) g^{n_1}$, hence taken together, after cancelling the factor g^{n_1}, the inequality

$$\text{(4)} \qquad m g^{n_0-1} < A_0 + 1, \quad \text{or else} \quad m g^{n_0-1} \leq A_0 .$$

Interpreted g-adically, this means the assertion of Theorem 1.

REMARK 1. As a consequence, the last digit-set A_0 must have exactly n_0 digits, so that in (1a) there are no digits 0 to insert in front of A_0.

REMARK 2. One confirms immediately, that with an n-digit number A also each number $A^{(r)}$ of the form

$$A^{(r)} = Ag^{(r-1)n} + \dots + Ag^n + A \quad (r = 1, 2, \dots) \tag{5}$$

satisfies the requests (1). Hence it suffices to determine the numbers A with a *minimal digit-number* $n = n_1 + n_0$.

§ 3. Sufficiency of the necessary condition.

THEOREM 2. *Leaving out of regard the trivial case of the multiplicator* $m=1$, *the necessary condition* (2) *is also sufficient for the existence of numbers A satisfying the requests* (1).

More exactly, to a given n_0-digit number A_0 as last digit-set and a multiplicator m with $1 < m \leq a_{n_0-1}$ there is exactly one minimal solution A of (1), *with the chain of coordinated solutions* (5) *arising from it by periodical repetition.*

The unique minimal solution A has the digit-number

$$n = \text{ord. } g \text{ mod. } \frac{mg^{n_0} - 1}{(A_0, mg^{n_0} - 1)}.$$

PROOF. As is clear from (3), the request (1b) may be expressed in the following form:

$$(g^{n_1} - m) A_0 = (mg^{n_0} - 1) A_1. \tag{6}$$

Here n_0 and A_0 are given, n_1 and A_1 to be found. For n_1 alone this means the request $(g^{n_1} - m) A_0 \equiv 0 \text{ mod. } (mg^{n_0} - 1)$, or else

$$g^{n_1} - m \equiv 0 \text{ mod. } \frac{mg^{n_0} - 1}{(A_0, mg^{n_0} - 1)}.$$

Since g is coprime to that congruence module, this is equivalent to the following request:

$$g^n \equiv mg^{n_0} \text{ mod. } \frac{mg^{n_0} - 1}{(A_0, mg^{n_0} - 1)} \quad \text{for } n = n_1 + n_0 \text{ instead of } n_1,$$

which in turn reduces to

$$(7) \qquad g^n \equiv 1 \bmod. \frac{mg^{n_0}-1}{(A_0,\, mg^{n_0}-1)}\,.$$

This request is satisfied minimally by the order n given in the assertion of Theorem 2.

For satisfying the basic request (1), this helps only when that order satifies $n > n_0$. The latter is indeed true under the restriction $m>1$, made in Theorem 2, as will be confirmed by the following indirect argumentation.

For brevity let $(A_0, mg^{n_0}-1)=t$ denote the above greatest common divisor and $(mg^{n_0}-1)/t=M$ the above congruence module. Suppose $n \leq n_0$, hence $n_0 = n+\nu$ with $\nu \geq 0$. Then, besides the above congruence (7), viz., $g^n \equiv 1 \bmod. M$, there would hold trivially also $mg^{n+\nu} \equiv 1 \bmod. M$. One would have then $mg^{\nu} \equiv 1 \bmod. M$, and since $m>1$, surely $m>M$. But this leads to a contradiction, because by definition

$$t \mid A_0 \text{ and } A_0 < g^{n_0}, \quad \text{hence} \quad t < g^{n_0},$$

and therefore in truth

$$M = (mg^{n_0}-1)/t > (mg^{n_0}-1)/g^{n_0} > m - 1/g^{n_0}, \text{ hence } M \geq m.$$

It being thus confirmed that for $m>1$ the order n, defined in Theorem 2, satisfies $n>n_0$, and hence by $n=n_1+n_0$ determines a possible digit-number n_1, the thread of the proof of Theorem 2, interrupted after (7), can be taken up again.

According to (6), the initial digit-set A_1 is uniquely determined as

$$(8) \qquad A_1 = A_0\,(g^{n_1}-m)/(mg^{n_0}-1) = A_0\,(g^{n-n_0}-m)/(mg^{n_0}-1).$$

This number A_1 has indeed exactly n_1 digits, as is shown by the following estimates.

On the one hand, according to the necessary solvability condition in the two forms (4) and (2), respectively:

$$A_1 > A_0\,(g^{n_1}-m)/mg^{n_0} \geq (g^{n_1}-m)/g > (g^{n_1}-g)/g = g^{n_1-1}-1,$$

hence $A_1 \geq g^{n_1-1}$.

On the other hand, the restriction $m>1$ implies $mg^{n_0}-1>g^{n_0}$, and hence

$$A_1 < A_0\,(g^{n_1}-m)/g^{n_0} < g^{n_1}-m < g^{n_1}.$$

The requested complete number A is then uniquely determined as

$$A = A_1 g^{n_1} + A_0 = A_0 (g^n - 1)/(mg^{n_0} - 1). \tag{9}$$

It has obviously exactly n digits. Its form (9) shows that passing to the multiples rn of the order n comes back to multiplying by the factors

$$(g^{rn} - 1)/(g^n - 1) = g^{(r-1)n} + \dots + g^n + 1,$$

or else, the periodic structure (5) of the non-minimal solutions.

This finishes the proof of Theorem 2.

§ 3. Examples.

The necessary solvability condition (2) shows that, apart from the trivial case $m=1$, the basis $g=2$ is out of the question.

It may suffice here to give some examples for the accordingly smallest possible basig $g=3$. By condition (2), here only $m=2$ is possible, and given the number

$\begin{Bmatrix} n_0 = 1 \\ n_0 = 2 \end{Bmatrix}$ of the last digit-set A_0,

only with $\begin{Bmatrix} A_0 = 2 \\ A_0 = 21 \text{ or } 22 \end{Bmatrix}$ (triadically).

According to (7), the uniquely determined solutions A for these three cases have the digit-numbers

$$n = \text{ord. } 3 \text{ mod. } \begin{Bmatrix} \dfrac{2 \cdot 3^1 - 1}{(A_0, 2 \cdot 3^1 - 1)} \\ \dfrac{2 \cdot 3^2 - 1}{(A_0, 2 \cdot 3^2 - 1)} \end{Bmatrix},$$

hence (the numerators being prime numbers, the denominators being 1)

$$n = \text{ord. } 3 \text{ mod. } \begin{Bmatrix} 5 \\ 17 \end{Bmatrix}, \text{ i. e.,} \quad n = \begin{Bmatrix} 4 \\ 16 \end{Bmatrix}.$$

For the first digit-sets A_1 this furnishes the digit-numbers

$$n_1 = \begin{Bmatrix} 3 \\ 14 \end{Bmatrix}.$$

The coordinated minimal solutions, to be constructed according to formulae (8), (9), are then found by easy calculation to be

$$A = \begin{Bmatrix} 1012 \\ 1030\ 1001\ 1202\ 1221, \ 1002\ 0100\ 1120\ 2122\ \text{resp.ly} \end{Bmatrix} \text{ (triadically).}$$

BIBLIOGRAPHY

[1] L. Bernstein: *Multiplicative Twins and Primitive Roots*, Math. Zeitschr. 105 (1968), 49-58.

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