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HALTER-KOCH UNITS FROM THE PERIODICITY OF ACF ALGORITHM

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Abstract

The author uses the periodicity of a modified Jacobi-Perron Algorithm, extended over the complex numbers, called ACF, in order to derive Halter-Koch units.

Halter-Koch considered the field determined by

$$P(x) = \prod_{j=1}^{r_1} (x-d_j) \prod_{j=r_1+1}^{r_1+r_2} (x - z_j)(x - \bar{z}_j) - d$$

with $r_1 \geq 0$, $r_2 \geq 0$, $n = r_1 + 2r_2 \geq 3$; $d, d_j \in Z$, $d \neq 0$; and for $|d| \geq 1$, ej are:

$$e_j = \begin{cases} d(w - d_j)^n, & 1 \leq j \leq r_1 \\ d^2((w - z_j)(w - z_j))^{-n}, & r_1+1 \leq j \leq r_1 + r_2 \end{cases}$$

for $|d| = 1$, n is replaced by 1.

To derive the units e_j, Halter-Koch did not use an algorithm. For the first time Bernstein using his Zero Algorithm derived Halter-Koch units from its periodicity.

In this paper, the author will use the above mentioned ACF Algorithm to derive the same units, and it has the advantage that all so far known results in the theory of units can be derived by means of a unified periodic ACF Algorithm.

Introduction

Hilbert's "Zahlbericht" to find the group of units in Algebraic Number fields form the periodicity of a common algorithm is still an open question.

The first attempt to satisfy Hilbert's demand was by Hasse and Bernstein [7,8] in 1965. They considered a modification of Jacobi-Perron Algorithm

(JPA). JPA is very difficult to prove periodic. Lagrange direction, periodic implies algebraic components for the starting vector is completely proved but Euler direction, algebraic components for the starting vector implies a periodic JPA is still an open difficult question. Bernstein devoted his entire life to work on this difficult problem and the periodicity of JPA remained an unsolved problem.

They considered Q(w) where

$$w = \sqrt[n]{D^n + d} \quad \text{with} \quad P(x) = \left(\prod^{n} (x^n - D_i)\right) - d$$

$D_i \in N$, d|D and the starting vector

$$a^{(0)} = ((w - D_1) \dots (w - D_{m-1}), \dots, (w - D_1)(w - D_2), (w - D_2))$$

with $b^{(0)}$ the companion vector as $a^{(0)}$ evaluated at $w = D_1$.

For $w = \sqrt[n]{D^n + d}$ they proved that JPA of $a^{(0)}$ is periodic if $D \geq (n - 2)d$ and for $w = \sqrt[n]{D^n - d}$ if $D \geq 2(n - 1)d$, $n \geq 3$.

In fact JPA of $a^{(0)}$ is purely periodic and the length of the period is n(n - 1).

Euler direction is still open since there are bounds on D and d must divide D.

No periodicity for $w = \sqrt[5]{12^5 + 6}$ since $D \not\geq 6(5 - 2) = 18$.

From the periodicity of their modified JPA, Hasse and Bernstein got units in Q(w), $e = \prod_{i=\ell}^{l+(m-1)} a_{n-1}^{(i)}$ for periodic JPA and $e = \prod_{i=0}^{m-1} a_{n-1}^{(1)}$ for purely periodic JPA, and Hasse and Bernstein units in Q(w) are

$$\begin{cases} e_s = \dfrac{w^s - D^s}{(w - D)^s} \; ; s > 1, s|n; |d| > 1 \\ e_s = w^s - D^s; \; s \geq 1, s|n; |d| = 1. \end{cases}$$

This is a fantastic result since for the first time units in algebric fields are obtained from a periodic algorithm.

Bernstein [6] used another modification of JPA called Zero Algorithm (ZA) and from its periodicity he derived Halter-Koch units using an Algorithm.

1. An algorithm over complex numbers field (ACF)-Preliminaries

The author [1] used a modification of JPA where for the first time not only the real numbers but also complex numbers were considered. This is the main distinction between Hasse and Bernstein modification of JPA and the author's modification of JPA. We consider the starting vector

$$(1.1)\qquad \begin{cases} a^{(0)} = (f_{1,n-1}(w), f_{1,n-2}(w), \ldots, f_{1,2}(w), f_{2,2}(w)) \\ f_{i,k}(w) = \prod_{s=i}^{k} (w - D_s) \\ f_{i,i}(w) = w - D_i, \ 1 \le i \le k \le n. \end{cases}$$

with

$$(1.2)\qquad \begin{cases} P(x) = \left(\prod_{i=1}^{k} (x^{s_i} - D_i^{s_i}) - d \right) \\ k \ge 2, \ s_i \ge 1, \ D_i \in N, \ d | D_i; \\ d \in Z, \ i = 1, 2, \ldots, k; \ |d| \ge 1; \\ 0 < D_1 < D_2 < \ldots < D_k. \end{cases}$$

P (x) is irreducible over the field of rationals [1] and has at least one real root w of the form

$$(1.3)\qquad w = \sqrt[n]{D^n + d}, \ d|D, \ D \in N, \ d \in Z, \ n \ge 2.$$

The starting companion vector $b^{(0)}$ of the starting vector $a^{(0)}$ is again the evaluation of $f_{1,i}$ at D_1. It could be at any D_i where D_i are complex numbers and $\{D_1, D_2, \ldots, D_n\}$ is any permutation of $\{D, \rho D, \ldots, \rho^{n-1} D\}$

We proved [1] that ACF is purely periodic with length of the period $m = n(n - 1)$. All the properties of JPA are preserved for ACF and all Hasse-Bernstein results for JPA are true for ACF. The advantage is that ACF eliminates the bound on D when $d < 0$ and $d > o$ in Hasse-Bernstein case and only the restriction $d|D$ remains.

For $w = \sqrt[5]{12^5 + 6}$ JPA is not periodic, but ACF is periodic since only $6 \mid 12$ is required.

Theorem 1. A unit in the field $Q(w,\rho)$ is given by the expression:

(1.4) $\quad e = d^{-(n-1)} ((w-D_2)(w-D_3) \dots (w-D_n))^n$

We choose the s_i units in $Q(w,\rho_i)$

$$
(1.5) \qquad \begin{cases} e_{i,0} = \dfrac{(w - D_i)^n}{d}, \\ e_{i,1} = \dfrac{(w - \rho_i D_i)^n}{d}, \\ e_{i,2} = \dfrac{(w - \rho_i^2 D_i)^n}{d}, \\ \dots\dots\dots \\ e_{i,s_i - 1} = \dfrac{(w - \rho_i^{s_i - 1} D_i)^n}{d}. \end{cases}
$$

and if we multiply all the units in (1.5) we obtain the unit

$$
(1.6) \qquad e_i = \frac{(w^{s_i} - D_i^{s_i})^n}{d^{s_i}} \qquad (i = 1, \dots, k)
$$

which are the k units in $Q(w)$. The units (1.4) were obtained by ACF. The proofs and the results here follow the proofs and the results of [1].

2. The statement of the problem

Halter-Koch considered the field determined by

$$(2.1)\qquad P(x) = \prod_{j=1}^{r_1} (x - d_j) \prod_{j=r_1+1}^{r_1+r_2} (x - z_j)(x - \bar{z}_j) - d$$

with

$$\begin{cases} r_1 \geq 0,\ r_2 \geq 0,\ n = r_1 + 2r_2 \geq 3,\ d,\ d_j \in Z, \\ d \neq 0;\ d_1 > d_2 > \dots > d_{r_1} \end{cases}$$

z_j are integral, complex solutions, $\bar{z}_j$ their conjugates, $d|d_i - d_j$, $d|d_i - z_j$, $d|z_i - z_j$, $d|z_i - \bar{z}_j$ for all possible indices i,j; if $r_1 = 3$, $r_2 = 0$, $|d| = 2$, then additionally $d_1 - d_2 \geq 4$ or $d_2 - d_3 \geq 4$.

Halter-Koch proved that P(x) has exactly r_1 (different) real zeroes and exactly r_2 (different) pairs of complex conjugate zeroes. To prove this, he needs the additional restrictions

$$(2.2)\qquad |d_i - d_j|,\ |d_i - z_j|,\ |z_i - z_j|,\ |z_i - \bar{z}_j| \text{ all to be } \geq 2$$

for all possible indices i,j.

In $Q(w)$, $w = \sqrt[n]{D^n + d}$, $D \in N$, $d \in Z$, $P(w) = 0$, then a complete system of fundamental units consists of $r_1 + r_2 - 1$ elements.

Halter-Koch proves that for $|d| > 1$

$$(2.3)\qquad e_i = \begin{cases} d(w - D_i)^{-n}, & 1 \leq i \leq r_1 \\ d^2((w - z_i)(w - \bar{z}_i))^{-n}, & r_1 + 1 \leq i \leq r_1 + r_2 \end{cases}$$

are $r_1 + r_2$ units in Q(w) with $\prod_{j=1}^{r_1+r_2} e_j = 1$, and that any (different) $r_1 + r_2 - 1$ of them form a complete system of independent units in Q(w). For $|d| = 1$, the exponent n in (2.3) has to be replaced by 1.

In all of these results Halter-Koch did not make use of any algorithm.

3. The solution of the problem

Let denote

(3.1) $\{D_1, D_2, ..., D_n\} = \{d_1, d_2, ..., d_{r_1}, z_{r_1+1}, \bar{z}_{r_1+1}, ..., z_{r_1+r_2}, \bar{z}_{r_1+r_2}\}$

where $d_i, z_j, \bar{z}_j$ ($i = 1, ..., r_1$; $z_j, \bar{z}_j = r_1+1, ..., r_1 + r_2$) are given by (2.1) and we rewrite P(x) from (2.1) as follows

(3.2) $P(x) = (x - D_1)(x - D_2) \dots (x - D_n) - d.$

From here we can apply ACF and we obtain through periodicity of ACF the unit

$$e = \frac{\left(\prod_{i=1}^{n} (w - D_i)\right)^n}{d^{n-1}} = \frac{d}{(w - D_i)^n}.$$

$i = 1, 2, ..., r_1, r_1 + 1, ..., r_2, r_2 + 1, ..., 2r_2$

where $r_1 + 2r_2 = n$.

For $D_i = d_i$ $\quad i = 1, 2, ..., r_1$ we obtain $e_i = d(w - d_i)^{-n}$.

For $D_i = z_i$, $\quad i = r_1+1, ..., r_2$ we obtain $e_i^* = d(w - z_i)^{-n}$ and

for $D_i = \bar{z}_i$, $\quad i = r_1+1, ..., r_2$ we obtain $\bar{e}_i^* = d(w - \bar{z}_i)^{-n}$. But

$e_i = e_i^* . \bar{e}_i^* = d^2((w - z_i)(w - \bar{z}_i))^{-n}$ is also a unit and this completes the proof for Halter=Koch units in (2.3).

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