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FIBONACCI TRIPLES AND PYTHAGOREAN TRIANGLES OF EQUAL AREA

by

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ABSTRACT

Fibonacci triples (3, 5, 8) form two adjacent Fibonacci pairs (3, 5) and (5, 8) of successive Finobacci numbers, which are the solutions of diophantine equations of the form $a^2 \pm ab + b^2 = c^2$. The solutions of these Diophantine equations are related to the problem of Pythagorean triangles of equal areas.

Key Words and Phrases

Rational Pythagorean Triangles (abbr. RPT), Fibonacci Triples, n-dimensional Lucas Numbers, Adjacent Fibonacci pairs, Diophantine equations.

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1. Definitions and Previous Results

A triangle with sides a, b, c which are represented by a triple (a, b, c) of natural numbers will be called a Rational Pythagorean Triangle (abbr. RPT), if and only if there exists (u, v),

$$\left.\begin{array}{l}(u, v) \in N^2 - \{(0, 1)\},\ u > v, \text{ such that } a = u^2 - v^2\ b = 2uv \\ c = u^2 + v^2,\ a, b, c \in N - \{0\} = 1, 2, \ldots .\end{array}\right\} \quad (1.1)$$

Denote such triangle by RPT(u, v)

The area of the right triangle

(denoted by S(u, v)) is $S(u, v) = \frac{1}{2}\ ab = uv\,(u^2 - v^2)$.

The first who asked the question to find triplets RPT-s having equal areas was the great Diophantus [4] and Dickson [3] enlarged the topic.
Let D be a triangle with integral sides and $\hat{c} = 120°$ one of its angles .
Then, if c is the side opposite $\hat{c}$ and a, b the two adjacent sides of $\hat{c}$, we have by $c^2 = a^2 + b^2 - 2ab\cos\hat{c}$

$$\left.\begin{array}{l} a^2 + ab + b^2 = c^2 \\ a + b > c > b > a;\ a, b, c \in N - \{0\} \end{array}\right\} \quad (1.2)$$

and if c = 60° we have

$$\left.\begin{array}{l} a^2 - ab + b^2 = c^2 \\ b > c > a > 0;\ a, b, c \in N - \{0\} \end{array}\right\} \quad (1.3)$$

Baica [1] connected the solutions of those two diophantine equations with the areas of the triangles. The totality of solutions to $a^2 \pm ab + b^2 = c^2$ is given in parameter form by Hasse [5]. No explicite solutions of (1.2) and (1.3) were known.
Since (1.2) and (1.3) are homogeneous diophantine equations, with a proper linear transformation, they can be reduced to a simple diophantine equation which can be solved explicity.

Theorem 1 of [1] states that if a, b, c satisfy the equation $a^2 + ab + b^2 = c^2$, (where $a + b > c > b > a$, and $a, b, c \in N - \{0\}$, then the three triangles RPT(c, a), RPT(c, b), and RPT(a + b, c) all have the same area, namely $S = abc(a + b)$.

To find an infinite number of solutions to the equation $a^2 + ab^2 + b^2 = c^2$ (but not necessarily all), let $a = y - 1$, and $b = y + 1$. (1.4)

The substitution yields $c^2 - 3y^2 = 1$, which is Pell's equation (1.5)

There are infinitely many pairs of numbers (u_n, v_n) which satisfy Pell's equation $u_n^2 - 2v_n^2 = 1$ where

$$u_n + \sqrt{3}\ v_n = (2 + \sqrt{3}\ )^n,\ n = 0, 1, \ldots \quad (1.6)$$

From (1.6) we deduce

$$u_n = \sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n}{2i} 2^{n-2i}3^i; \quad n = 0, 1, \ldots$$

$$V_n = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n}{2i+1} 2^{n-1-2i}3^i; \quad n = 0, 1, \ldots \tag{1.7}$$

for n = 2m

$$u_{2m} = \sum_{i=0}^{m} \binom{2m}{2i} 2^{2m-2i}3^i; \quad m = 1, 2, \ldots$$

$$V_{2m} = \sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^{2m-1-2i}3^i; \quad m = 1, 2, \ldots \tag{1.8}$$

$$(u_0, v_0) = (1, 0)$$

or for n = 2m + 1

$$U_{2m+1} = \sum_{i=0}^{m} \binom{2m \div 1}{2i} 2^{2m+1-2i}3^i; \quad m = 0, 1, \ldots$$

$$U_{2m+1} = \sum_{i=0}^{m} \binom{2m+1}{2i+1} 2^{2m+1-2i}3^i; \quad m = 0, 1, \ldots \tag{1.9}$$

Some of these pairs are $(u_0, v_0) = (1, 0)$, $(u_1, v_1) = (2, 1)$, $(u_2, v_2) = (7, 4)$, $(u_3, v_3) = (26, 15)$, etc. (1.10)

Consider the pair $u_2 = c - 7$ and $v_2 = y = 4$ then a = 3, b = 5 and c = 7 and the triangles are

$$\left.\begin{aligned} RPT(7, \underline{3}) &= (40, 42, 58) \\ RPT(7, \underline{5}) &= (24, 70, 74) \\ RPT(\underline{8}, 7) &= (15, 112, 113) \end{aligned}\right\} \tag{1.11}$$

all these triangles (1.11) have area = 3.5.7.8 = 840.

Note that the three underlinded numbers are three consecutive Fibonacci numbers.

Theorem 2 in [1] states that if a, b, c satisfy the equation $a^2 - ab + b^2 = c^2$, $b > c > a > 0$, then there are the three triangles RPT(b, c), RPT(c, a) and RPT(c, b-a) all have the same area, namely $S = abc(b - c)$.

Using a similar method to find an infinite number of solutions let

$a = \frac{1}{2}(y + 1)$ and $b = y - 1$; $(y \geq 2)$ (1.4a).

Substituting yields $4c^2 - 3(y - 1)^2 = 4$ (y odd). If y is odd, then y - 1 is even and the equation may be divided by 4 to give:

$c^2 - 3\left(\frac{y - 1}{2}\right)^2 = 1$ which is again Pell's equation (1.5a)

Again the (u_n, v_n) pairs provide infinitely many solutions, where $u_n = c$ and

$v_n = \frac{y - 1}{2}$ or $y = 2v_n + 1$.

Consider again the pair (7, 4); then $c = 7$ and $y = 9$ so $a = 5$, $b = 8$, $c = 7$, and the triangles are:

$$\left.\begin{array}{l} RPT(8, 7) = (15, 112, 113) \\ RPT(7, 5) = (24, 70, 74) \\ RPT(7, 3) = (40, 42, 58) \end{array}\right\} \quad (1.11a)$$

(1.11a) are the same triples as in (1.11). Now we are asking whether two successive Fibonacci numbers would be solutions (a, b) of (1.2) or (1.3)?

Note that c does not have to be a Fibonacci number, but that a + b will be the third Fibonacci number involved in the RPT triplet.

The Fibonacci sequence with $F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$, $n = 1, 2, \ldots$ goes 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Here $(F_4, F_5) = (3, 5)$ is a solution of the Diophantine equation (1, 2).

(i. e.) $3^2 - 3 \cdot 5 + 5^2 = 7^2$ and $(F_5, F_6) = (5, 8)$ is a solution of (1.3)

(i. e.) $5^2 - 5 \cdot 8 + 8^2 = 7^2$.

In this paper we intend to look for more pairs of adjacent Fibonacci numbers as serving as solutions for (1.2) and (1.3).

2. Main Result

At a conference in April 1990, R. Pinch suggested using ϕ and $\overline{\phi}$ as the Fibonacci generators for this problem. Using standard notation for Fibonacci numbers;

$$\left.\begin{array}{l} \text{let } \phi = \dfrac{1+\sqrt{5}}{2} \quad \text{and} \quad \overline{\phi} = \dfrac{1-\sqrt{5}}{2} \\ \text{where } \phi + \overline{\phi} = 1 \text{ and } \phi\,\overline{\phi} = -1. \end{array}\right\} \qquad (2.1)$$

$$\text{Let } F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \overline{\phi}^n\right) \qquad (2.2)$$

(So the first Fibonacci number corresponds to $n = 1$)

Let $a = F_n$, $b = F_{n+1}$, and substitute it into

$$a^2 + ab + b^2 = c^2, \text{ getting } F_n^2 + F_nF_{n+1} + F_{n+1}^2 = c^2 \qquad (2.3)$$

Substituting the ϕ notation:

$$\frac{1}{5}\left(\phi^{2n} - 2\phi^n\,\overline{\phi}^n + \overline{\phi}^{2n}\right) + \frac{1}{5}\left(\phi^{2n+1} - \phi^{n+1}\,\overline{\phi}^n - \phi^n\,\overline{\phi}^{n+1} + \overline{\phi}^{2n+1}\right) +$$

$$+ \frac{1}{5}\left(\phi^{2n+2} - 2\phi^{n+1}\,\overline{\phi}^{n+1} + \overline{\phi}^{2n+2}\right) = c^2$$

Factor the $\frac{1}{5}$, rearrange, and use (2.1) to get:

$$\frac{1}{5}\left[\phi^{2n} + \overline{\phi}^{2n} + \phi^{2n+1} + \overline{\phi}^{2n+1} + \phi^{2n+2} + \overline{\phi}^{2n+2} + 2(-1)^{n+1} - 2(-1)^{n+1} - (-1)^n(1)\right] = c^2$$

Thus

$$\phi^{2n} + \overline{\phi}^{2n} + \phi^{2n+1} + \overline{\phi}^{2n+1} + \phi^{2n+2} + \overline{\phi}^{2n+2} \pm 1 = c^2 \qquad (2.4)$$

with -1 when n is even, $+1$ when n is odd.

Let L_n be a Lucas number of the form $L_n = \phi^n + \overline{\phi}^n$ (2.5)

Then we have (from 2.4)

$$L_{2n} + L_{2n+1} + L_{2n+2} \pm 1 = 5c^2 \tag{2.6}$$

By the recursive property at L_n, we have $L_{2n} + L_{2n+1} = L_{2n+2}$ (2.7)

$$\text{or } 2L_{2n+2} \pm 1 = 5c^2 \tag{2.8}$$

Solving: $L_{2n+2} = \dfrac{5c^2 \mp 1}{2}$

Use the ϕ notation to discover the relation between the Lucas number

$$L_n = \phi^n + \overline{\phi}^n \text{ and } F_n = \frac{1}{\sqrt{5}}(\phi^n - \overline{\phi}^n)$$

$$L_n^2 = \phi^{2n} + 2\,\phi^n\,\overline{\phi}^n + \overline{\phi}^{2n} = \phi^{2n} + 2(-1)^n + \overline{\phi}^{2n}$$

$$F_n^2 = \frac{1}{5}(\phi^{2n} - 2\phi^n\,\overline{\phi}^n + \overline{\phi}^{2n}) = \frac{1}{5}(\phi^{2n} - 2(-1)^n + \overline{\phi}^{2n})$$

$$L_n^2 - 2(-1)^n = \phi^{2n} + \overline{\phi}^{2n} = 5F_n^2 + 2(-1)^n$$

$$\text{So } L_n^2 = 5F_n^2 \pm 4 \tag{2.9}$$

with $+4$ when n is even, -4 when n is odd.

By (2.9) we have $L_{2n+2}{}^2 = 5F^2{}_{2n+2} + 4$ (+ since $2n + 2$ is even) and by (2.8)

$$\left(\frac{5c^2 \pm 1}{2}\right)^2 = 5F^2{}_{2n+2} + 4$$

It is fairly easy to see that $5F^2{}_{2n+2} + 4$ is always a perfect square, so let

$F = \sqrt{5F^2_{n+2} + 4}$. Then $\frac{5c^2 \pm 1}{2} = F$, or solving for c^2:

$$c^2 = \frac{2F \pm 1}{5}$$

This equation clearly has infinitely many solutions if F is allowed to be arbitrary, but for a Fibonacci solution, we must have $F = \sqrt{5F^2_{2n+2} + 4}$. So far, the only Fibonacci solution, we have discovered to yield a $\frac{2F \pm 1}{5}$ which is a perfect square is 55 (when n = 4, so this gives a = 3, b = 5).

Using a direct attack, and calculating $a^2 + ab + b^2$ for pairs of consecutive Fibonacci numbers, it can be shown that if $F_n^2 + F_nF_{n+1} + F^2_{n+1}$ is a perfect square, then n must be of the form 30m + k, where m is any integer ≥ 0 and k is 0, 4, 5, 10, 18, 23, 24, 28 or 29. (2.10)

This follows from observations about the last three digits of perfect squares which end in 1 or 9 (since $F^2_n + F_nF_{n+1} + F^2_{n+1}$ always ends with a 1, 3, 7 or 9: the actual pattern is 3, 7, 9, 9, 9, 7, 3, 1, 1, 1, 3, 7, etc.).

If ... abc is a perfect square with last three digits a, b, and c, where c is 1 or 9, then b is always even, and the relationship between a and b is as follows: If a is odd, then b is 2 or 6. If a is even, then b is 0, 4, or 8.

The last three digits of $F_n^2 + F_nF_{n+1} + F^2_{n+1}$ have a pattern which repeats with a period of 750; the only ones which could be a perfect square c^2 are the ones with n as indicated in (2.10).

A computer check up to n = 275 has not yet revealed any perfect squares, except of course for n = 4.

Our explorations have not yet yielded any final answer to the question of whether there are any adjacent Fibonacci numbers, other than 3 and 5, which can be used for a and b in the solution of the Diophantine equation $a^2 + ab + b^2 = c^2$. At this time it appears that there are not. We will continue the search.

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