(18)

ANALELE UNIVERSITATII „EFTIMIE MURGU" RESITA

Sesiune jubiliară

Un sfert de veac de la fondarea învăţământului superior la Reşiţa

FASCICOLA III

A. ELECTROTEHNICĂ ŞI SISTEME AUTOMATE
B. METALURGIE

ANUL III,
1996

ANALELE UNIVERSITATII „EFTIMIE MURGU" RESITA

Secțiunea

ELECTROTEHNICĂ ŞI SISTEME AUTOMATE

Coordonatori ,

Prof. dr. ing. Octavian Crivacucea ,
Prof. dr. ing. Tiberiu Mănescu ,
Ş.l. ing. Adrian Ghițiu

Cordoonare generală ,

Prof. dr. ing. Mircea Golumba ,
Conf. dr. Vasile Mircea Zaberca

UEM Resita

UN SFERT DE VEAC DE LA
FONDAREA INVATAMANTULUI SUPERIOR LA RESITA
25-26 octombrie 1996

BAICA'S SOLUTION OF FERMAT'S LAST THEOREM IN EUCLIDIAN, MODELS AND ALGORITHMS THE ~~TRANSACTION~~ TRANSITION FROM ABSTRACT TO APPLIED MATHEMATICS

Malvina Baica

ABSTRACT In this paper the author gives a description of Baica's General Euclidean Algorithm and its use to give complete solutions of famous unsolved problems in Number Theory and Algebraic Number Theory. Among them are the solutions of Fermat's Last Theorem, Hermite's Problem, Dirichlet's Problem, Hilbert's Universal Algorithm Periodicity Problem, Galois' Theory of Polynomials Problem and many others.

KEY WORDS AND PHRASES

EUCLIDEAN ALGORITHM (abbr. EA)

JACOBI-PERRON ALGORITHM (abbr. JPA)

BAICA'S ALGORITHM IN A COMPLEX FIELD (abbr. ACF)

(ACF) ALSO NAMED BAICA'S GENERAL EUCLIDEAN ALGORITHM (abbr. BGEA)

HASSE-BERNSTEIN MODIFICATION OF JPA (abbr. HBA)

FERMAT'S LAST THEOREM (abbr. FLT)

HILBERT'S UNIVERSAL ALGORITHM PERIODICITY PROBLEM (Abbr. HUAPP)

HERMITE'S PROBLEM (abbr. HERP)

DIRICHLET'S PROBLEM (abbr. DIRP)

GALOIS' THEORY OF POLYNOMIALS PROBLEM (abbr. GTPP)

EUCLIDEAN GEOMETRY (abbr. EG)

THE GEOMETRY OF THE ELLIPTIC CURVES (abbr. GEC)

EUCLIDEAN MODEL OR EUCLIDEAN VARIETY (abbr. EV)

THE VARIETY OF ELLIPTIC CURVES (abbr. VEC)

EULER-LAGRANCE THEOREM (abbr. ELT)

0. INTRODUCTION

In Mathematics, problems can be approached geometrically, algebraically or analytically, and proofs can be given in different mathematical varieties, which are defined in the modern mathematics as mathematical models.

We can associate to each geometry a corresponding algebra and from here we can say that to every Geometric Variety (GV) it corresponds an Algebraic Variety (AV), but the converse is not true.

As it can be seen, we can construct as many Geometries or Geometrical Models as we please. All that we need is to have the elements declared, to state the axioms and the definitions and to have consistency in our logic.

For example, in the (EG) the elements are points and the straight line. For Hyperbolic Geometry (HG) the elements are points and the lines are hyperbolas, for Elliptical Geometry (ELG) the elements are points and the lines are ellipses for Spherical Geometry (SG) the elements are points and the lines are the circles, and there are also many Projective Geometries (PG). Every Geometry has its corresponding associate Algebra. Only one geometry is the Euclidean Geometry (EG), the other geometries are NON EUCLIDEAN geometries. The geometries do not report to each other, but they all report to the Topology. Because of this, if you prove something in one geometry it may not be the same as in other geometries.

Let's look for the example at the V-th postulate in (EG) what is be-coming of it in the Hyperbolic Geometry (HG)?

Something nice can happen when the results are equivalent. The necessary transformation from Elliptic to Euclidean is required to show that those results in two distinct geometries are equivalent. But this is not enough once it is proved equivalent, in order to show that results are the same there is a need to provide the Galois' connection from Category Theory. In other words, the functor has to be provided.

1. BAICA'S GENERAL EUCLIDEAN ALGORITHM (BGEA)

In order to develop an Algorithm all you need is a starting vector and a transformation function T. Iterating T we obtain a sequence $\{T, T^2, \ldots, T^l, T^{l+1}, \ldots, T^m, T^{m+1}, \ldots\}$

If $T^l = T^m$, then the algorithm of the starting vector is called periodic with the length of the preperiod, or tail, l and the length of the period m.

We start with the well known and powerful Euclidean Algorithm (EA). Another interpretation of (EA) which leads to the continued fraction is:

Let the starting vector be $a^{(0)} = (a_1^{(0)}) \in R^1$ and a transformation function which is the greatest integer function $[a_1^{(0)}]$ as a companion vector $b^{(0)} = [a_1^{(0)}] = (b_1^{(0)})$; then the recursive transformation

$$a^{(v+1)} = (a_1^{(v)} - b_1^{(v)})^{-1} = \frac{1}{(a_1^{(v)} - b_1^{(v)})}$$

applied to these vectors become a sequence $\{a^{(v)}\}$, $v = 0, 1, \ldots$; which is called the continued fraction interpretation of (EA). Because of the periodicity of (EA) many difficult problems which were still open in fields of higher degree can be solved in quadratic fields. For example, by using this algorithm it is easy to prove that every rational number, a/b, can be represented as a finite continued fraction or by a finite sequence.

In 1737, Euler proved that every real quadratic irrational can be represented by an infinite periodic continued fraction or by a periodic (EA) sequence development. The converse was proved by Lagrance in 1770. Of course, if the number is not a quadratic irrational, but is a real algebraic number of higher degree or a transcendental real number, then its development by the (EA) cannot be periodic.

In 1839, Hermite [16], in one of his letters to Jacobi, challenged Jacobi to find an algorithm to develop irrationals of any degree into periodic sequences. But it was only after thirty years of frustration that Jacobi [17], in 1869 extended (EA) methods to successfully represent some cubic irrationals by means of simple continued fraction.

Then in 1907, Perron [19], [20], generalized the work of Jacobi. This generalization is known as the Jacobi-Perron Algorithm (JPA). In its general form, as defined by Jacobi for $n = 3$ and by Perron for any $n \geq 2$, an application of the (JPA) starts with the definition of any initial vector, $a^{(0)} = (a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-1}^{(0)}) \in R^{n-1}$, $n \geq 2$ the components of which are algebraic numbers. By use of the greatest integer function a "companion vector"

$b^{(0)} = (b_1^{(0)}, b_2^{(0)}, \ldots, b_{n-1}^{(0)}) \in R^{n-1}$ with

$b_i^{(0)} = [a_i^{(0)}]$, $(i = 1, 2, \ldots, n-1)$ is defined. A recursive transformation

$$a^{(v+1)} = (a_1^{(v)} - b_1^{(v)})^{-1} \quad (a_2^{(v)} - b_2^{(v)}, \ldots, a_{n-1}^{(v)} - b_{n-1}^{(v)}, 1)$$

is constructed and applied to these vectors. Then the sequence

$\{a^{(v)}\}$, $v = 0, 1, \ldots$; is called JPA

For good choices of the starting vector $a^{(0)}$ and for the transformation, the iteration of the transformation becomes periodic, that is, the transformation cycles around a finite set of vectors. In the instance the (JPA) is said to be periodic, and the results lead to the (JPA) periodic representation of higher degree irrationals. The difficulties associated with this work are many. Jacobi's results were confined to a few numerical examples in a cubic field, where Jacobi exhibited periodic developments for $\sqrt[3]{2}$, $\sqrt[3]{3}$ and $\sqrt[3]{5}$. Perron generalized the method to apply to irrationals of any degree, but since the choices of a starting vector and transformation are difficult to make, he was also limited to a few periodic developments of higher degree irrationals. Those results were to prove an Euler direction for higher degree irrationals. Perron was more successful in showing that if a development is periodic then the components of the initial vector are algebraic numbers. This latter result was general, with this proving completely Lagrange direction for higher degree irrationals.

Advances were slow and difficult, but in 1873 Bachman proved results for other cubic irrationals using the (JPA); results that were accompanied by many restrictions. With this work on Hermite's Problem progress came to a halt, because of the failure of the (JPA) to produce new numerical results, that is, additional cases in which the transformation becomes periodic were not achieved. Perron and all others recognized that the usual choices for a starting vector were too limited. No further progress occurred on these problems until Hasse and Bernstein turned their attention to them in 1965, and made a broader approach to the periodicity problem associated with the (JPA). Hasse and Bernstein started with an algebraic extension of the rational numbers, Q(w), where w takes the form $w = \sqrt[n]{D^n + d}$ with

$$P(x) = \prod_{i=1}^{n} (x^n - D_i^n) - d, \ d \in Z, \ D_i \in N \text{ and } d \mid D.$$

$$a^{(0)} = ((w - D_1)(w - D_2) \ \dots \ (w - D_{n-1}), \ \dots, \ (w - D_1)(w - D_2),$$

$(w - D_1))$ with $b^{(0)} = a^{(0)}(D_1)$. They showed [13], [14], that certain restrictions on D and d led to a (JPA) that was purely periodic (that is that the length of the preperiod is zero). For $d > 0$ they proved that (JPA) of $a^{(0)}$ is purely periodic when $D \geq (n - 2)d$, $d \mid D$ and $n \geq 3$. For $d < 0$ the sequence is also purely periodic when $D \geq 2(n - 1)d$, $d \mid D$ and $n \geq 3$. With these conditons, the length of the period is $n(n-1)$.

For this approach the periodicity remains an open problem since there are bounds on D and the restriction $d \mid D$ must hold. For example, no periodicity for $w = \sqrt[5]{12^5 + 6}$ can be proved under (HBA) restrictions since $12 \not\geq (5 - 2)6 = 18$.

The Hasse and Bernstein results were limited by their choices of w as real numbers. It should be noted that Hasse and Bernstein were not interested in Hermite's problem in spite of the fact that their results had a strong relation to that problem. Specifically, they did not realize that the periodicity of the algorithm leads to a solution of Hermite's Problem for some real algebraic number w. They were interested in the Dirichlet's Problem.

In 1980, Baica [1] defined a modification of the (JPA) that used the Hasse and Bernstein initial vector, but was not restricted to the real numbers. For the first time the complex numbers were considered.

The only differences in the definitions stated alone are that the D_i's are now complex numbers. An immediate consequence of this extension is that the bounds on D in the (HBA) are now eliminated and only the divis--ibility condition, $d|D$, remains. Returning to the example cited above, it can now be seen that $w = \sqrt[3]{12^3 + 6}$ has a periodic development, only $6|12$ is required. Baica named her Algorithm, the Algorithm for Complex Numbers (ACF) and later she named (ACF) to be Baica's General Euclidean Algorithm (BGEA). All of the previous results of the (JPA) and all of the (HBA) results are consequences of (BGEA). In conclusion, Baica proved that all of the real numbers, and, for the first time all the complex numbers of the form w with $d|D$, have a periodic (BGEA) sequence development. This is the solution of Hermite's problem. From the fact that (BGEA) is not always periodic for $n \geq 3$ it follows that not all higher degree irrationals have a periodic (BGEA) sequence development.

All of the quadratic irrationals do have a (BGEA) periodic sequence development since, for n = 2, (BGEA) becomes (EA). This is the justification for naming (ACF) as (BGEA). (EA) and (BGEA) are always periodic for n = 2, but (BGEA) is not always periodic for $n \geq 3$ the restriction $d|D$ cannot be removed in proofs of the perodicity of (BGEA).

It is known the (EA) is periodic and as a consequence of its periodicity many difficult problems in Number Theory (NT) are solved in quadratics (n = 2). The same problems remained unsolved in n dimensions and this made Hilbert demand an Universal Algorithm from whose periodicity to solve all of the open problems in n dimensions, which are solved in quadratics from the periodicity of (EA) under the form of continued fractions. Let's call this Hilbert's Universal Algorithm Periodicity Problem (HUAPP). Logicians proved that Hilbert's <u>DREAMED</u> periodic

Algorithm does not exist. Therefore (BGEA) is an explicit proof of (HUAPP).

As well (BGEA) now completely solves Hermite's problem [6]. That is for every $w = \sqrt[n]{k}$ where $k = D^n + d$ and $d|D$ we have a (BGEA) periodic development which is an equivalent of (ELT) for nth degree irrationals.

The restricted periodicity of (BGEA) also completely solves Dirichlet's problem [1, 4, 8, 9]. That is, we can find the group of units in algebraic number fields Q(w) for all $w = \sqrt[n]{k}$, $k = D^n + d$ if $d|D$. Since Galois' Theory of Polynomials Problem is related with the group of units in Algebraic Number Fields Q(w) once that Dirichlet's problem is solved this will solve (GTPP)

It is well known that $x^2 + y^2 = z^2$ have integral solutions and this is an immediate consequence that (EA) is periodic. Hilbert was the one who related quadratics with the periodicity of (EA). Having the tool now, (BGEA) produce the generalization by induction on the dimension of (BGEA) and this proves that $x^n + y^n = z^n$ does not have integral solutions for $n \geq 3$ since (BGEA) is not always periodic of $n \geq 3$. That is known as (FLT). The conjecture known as (FLT) was stated by Fermat on the margin of his copy of Bachet's translation of Diophantus, at the side of Problem 8 of Book 2. "To divide a given square number into two squares." Fermat's marginal note reads, "To divide a given cube number into two cubes, a fourth power, or in general, any power whatever, into two powers of the same denomination above the second is impossible and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it."

This problem has baffled the best mathematicians for nearly 350 years. Therefore, the restricted periodicity of (BGEA) for $n \geq 3$

implies Fermat. In [2] Baica proved (FLT) in Euclidean (EV) as it supposed to be and which is closer with Fermat's heart. In 1995 an in-house publication in the Annals of Mathematics at Princeton accepted a proof of (FLT) in the (GEC) and its corresponding Algebra, which was overwhelmingly embraced by the American Mathematical Society The proof is 129 pages long. Considering the Euclidean character of FLT it should not be expected to be proved using the (GEC). Its proof in the (GEC) may not be equivalent to the results in (EG). The equivalency was not given yet. Of the Elliptic curves the required transformation (isomorphism) is from Elliptic to Euclidean and it seems to be more complicated, and it may not exist. The functor from Category Theory was not provided. In this case, the proof of FLT in the (GEC) will be much longer then it is now (129 pages) and will exceed the margins a lot more. Baica's proof of (FLT) in (EV) is the work of Euclid, Jacobi, Perron, Gauss, Euler, Hermite, Hilbert, Derichlet, Hasse, Bernstein and Baica put together. All of those great mathematicians before me ultimately were looking to solve (FLT) and historically they paved the way for me to finish the final step in its proof. Baica's solution of (FLT) is the evolutionary development of the algorithms of Jacobi, Perron, Hasse-Bernstein and Baica, and it is the only solution provided in Euclidean so far. The tool is (BGEA) and I got the solution putting together all related work from the History of Mathematics starting with Euclid up to our time. The opposite groups [21] who support the solution (?) of (FLT) in the (GEC) very "nicely" called all of those great mathematicians before Wiles as "cranks" and they want "to get the cranks off <u>our</u> backs." <u>Our</u> means <u>theirs</u>. As we see now those great mathematicians from the History of Mathematics the "cranks" are not off their backs yet, and the role of the great mathematicians who led the author to the proof of (FLT) in [2] should not be undermined.

The application of (BGEA) does not stop here. In many other published papers the author have extended the application of the periodicity of (BGEA) for solutions of very complicated diophantine equations [7]. She has developed very complicated combinatorial identities [3, 7], and recently, she used it to find the sums of some infinite series. For the first time she emphasized the importance of the periodicity of an algorithm as a tool to prove something in Mathematics.

In conclusion, (BGEA) is a very powerful algorithm when it becomes periodic. It is as competitive in n dimensions as it is (EA) in quadratics.

The (BGEA) will dominate mathematics for higher dimension fields over the years to come, exactly as (EA) dominated mathematics for quadratic fields for so many years in the past.

In the modern era the role of Modeling Theory in solving application problems is tremendously increasing.

You can generate algebraic, geometric, analytic, statistical, computational, and probabilistic models. Statistical models are very well known to be used in the decision making theory. The theory of Algorithms plays a big role already in the Computer Science now. The mathematical space models play a big role in Mechanical Physics and space research today. (BGEA) solves up to its restricted periodicity everything that were open problems in the (EV), or classical Number Theory and Algebraic Number Theory.

The future of Applied Mathematics will be to invent Mathematical

Models and Algorithms in order to solve the necessary application problems and new models and new algorithms will perform the transaction from Abstract to Applied Mathematics.

BIBLIOGRAPHY

[1] M. BAICA, An Algorithm in a complex field (ACF) and it applications to the calculation of units, Pacific J. Math. (1) 110 (1984) p. 21-40.

[2] M. BAICA, Baica's General Euclidean Algorithm and the solutions of Fermat's Last Theorem, NNTDM 1, (1995) 3, p. 120-134.

[3] M. BAICA, Some new combinatorial identities derived from units in Algebraic Number Fields, Discrete Mathematics 54 (1985) p. 133-141.

[4] M. BAICA, Hilbert's demand for the disclosure of units in Algebraic Number fields, Bull. Numb. Theory vol. XVI (1992) pg. 149-163.

[5] M. BAICA, Approximation of irrationals, Internat. J. Math. Sci. (2) 8 (1985) p. 303-320.

[6] M. BAICA, Hermite's Problem from the periodicity of (ACF) Algor -ithm, Bull. Numb. Theory, vol. XIV (1990) p. 57-67.

[7]. M. BAICA, Diophantine equations and identities, Internat. J. Math. and Math. Sci. vol. 8, no. 4 (1990) p. 755-777.

[8] M. BAICA, More units from the periodicity of an Algorithm, Bull. Numb. Theory. vol. XII (1988), p. 81-89.

[9] M. BAICA, Halter-Koch units from the periodicity of (ACF) Algorithm. Bull. Numb. Theory, vol. XIII (1989), p. 81-89.

[10] M. BAICA, Pythagorean Triangles of equal areas, Internat. J. Math. Sci. (11) 4 (1988) p. 769-780.

[11] M. BAICA, The Euclidean Character of the Fermat's Last Theorem, accepted for publication.

[12] M. BAICA, More Explanations about Baica's proof of Fermat's Last Theorem, accepted for publication.

[13] L. BERNSTEIN AND H. HASSE, Einheitenberechnung mittels des Jacobi-Perronschen Algorithmus. J. Reine. Angew, Math. 218 (1965), p. 51-69.

[14] L. BERNSTEIN, The Jacobi-Perron Algorithm, its Theory and Applications, Springer, Berlin-Heidelberg-New York, Zeit. Notes. Math. vol. 207, 1971.

[15] L. BERNSTEIN, Representation of $\sqrt[n]{D^n - d}$ as a periodic continued fraction by Jacobi's Algorithm, Math. Nachr. 29 (1965) p. 179-200.

[16] C. H. HERMITE, Letter to C. G. Jacobi, J.f.d. Reine. Angew. Math 40 (1939) p. 286.

[17] C. G. J. JACOBI, Allgemeine Theorie der kettenbruchaehnlichen Algorithmen, in velchen jede Zahl aus drei vorhergehenden gebildet wird, J.f.d. reine Angew. Math. 69 (1969), p. 29-64.

[18] H. LONDON and R. FINELSTEIN, On Morell's Equation. Bowling Green State University Press, Bowling Green 1973.

[19] O. PERRON, Grundlagen fuer eine Theorie des Jacobischen Kettenbruchalgorithmus, Math. Ann. 64 (1907), p. 1-76.

[20] O. PERRON, Ein neues Konvergenskriterium fur Jacobi-Ketten 2. Ordnung Arch. Math. Phys. (Reine 3) 17 (1911), p. 204-211.

[21] A. J. VAN DER POORTEN, Remarks on Fermat's Last Theorem, Macquarie Number Theory Reports from: Austral. Math. Soc. Gazette 21 (1994), p. 150-159.

MALVINA BAICA
Department of Mathematics and Computer Science
The University of Wisconsin
Whitewater, WI 53190 U.S.A.

ERATA

la cuprins ;