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GENERAL EUCLIDEAN ALGORITHM (BGEA) AND SUMS OF SOME INFINITE SERIES

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ABSTRACT. In this paper the author will use some previous results from her (ACF) Algorithm, now named Baica's General Euclidean Algorithm (BGEA), in the theory of units in algebraic number fields. Starting with a unit $e = \frac{(w-D)^n}{d}$ from (BGEA) and e^{-1} power of this unit in $Q(w)$, $w = \sqrt[n]{D^n + d}$, $D \in N$, $d \in Z$ and $d|D$, we will evaluate the sums of some infinite series. In one of her other papers [2], the author used a similar method to find new combinatorial identities.

Key words and Phrases:	Abbreviation:
BAICA'S ALGORITHM IN A COMPLEX FIELD	(ACF)
JACOBI - PERRON ALGORITHM	(JPA)
(ACF) ALSO NAMED BAICA'S GENERALIZED EUCLIDEAN ALGORITHM	(BGEA)

1.Introduction. The (ACF) [1] algorithm is an extension of the (JPA) over the complex numbers, and it is a very powerfull algorithm when it is periodic. The Euclidean Algorithm is a particular case of the (ACF), now named the (BGEA). Because of the periodicity of the Euclidean Algorithm, many difficult problems still open in higher fields are solved in the quadratic fields.

In this paper we will use some previous results from the (BGEA) in the theory of units in algebraic number fields [1].

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Starting with a unit $e=\dfrac{(w-D)^n}{d}$ from (BGEA) and powers of this unit in $Q(w)$, $w=\sqrt[n]{D^n+d}$, $D\in N$, $d\in Z$, $d|D$ and $w^n=m$, we will evaluate the sums of some infinite series. In one of my previous papers [2] I used a similar method to find new combinatorial identities.

2.The statement of the problem

<u>Theorem</u> If $e=\dfrac{(w-D)^n}{d}$ in $Q(w)$, $w=\sqrt[n]{D^n+d}$, $D\in N$, $d\in Z$, $d|D$ and $w^n=m$.

$$(1.1)\quad A=\begin{pmatrix} s_{0,1} & ms_{n-1,1} & \cdots & ms_{2,1} & ms_{1,1} \\ s_{1,1} & s_{0,1} & \cdots & ms_{3,1} & ms_{2,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n-2,1} & s_{n-3,1} & \cdots & s_{0,1} & ms_{n-1,1} \\ s_{n-1,1} & s_{n-2,1} & \cdots & s_{1,1} & s_{0,1} \end{pmatrix}$$

where

$$(1.2)\quad \begin{cases} s_{0,1}=\dfrac{m+(-1)^n D^n}{d} \\ s_{i,1}=(-1)^{n-i}\dbinom{n}{n-i}\dfrac{D^{n-i}}{d} \end{cases}$$

$n\in N$, $d\in Z$, $D\in N$, $d|D$, $n>2$, $m=D^n+d$

then

$$(1.3)\quad \begin{cases} \displaystyle\sum_{j=0}^{\infty}\binom{(j+1)n-1}{jn}\left(\frac{D^n}{m}\right)^j=\frac{m}{d}|A_1| \\ \displaystyle\sum_{j=1}^{\infty}\binom{(j+1)n-i-1}{jn}\left(\frac{D^n}{m}\right)^j=\frac{m}{d}|A_{i+1}| \end{cases}$$

$$(i=1,2,\ldots,n)$$

where $|A_1|$ is $|A|$ with the first column replaced by the vector $(1,0,\ldots,0)$ and $|A_{i+1}|$ is $|A|$ with the i-th column replaced by the vector $(1,0,\ldots,0)$.

Proof

Suppose

$$(1.4)\quad \begin{cases} e = s_{0,1} + s_{1,1}w + \cdots + s_{n-1,1}w^{n-1} & ;\ s_i \in Z \\ e^{-1} = t_{0,1} + t_{1,1}w + \cdots + t_{n-1,1}w^{n-1} & ;\ t_i \in Z \\ \qquad i = 0,1,\ldots,n-1 & \end{cases}$$

We perform ee^{-1} reducing the powers of w , knowing that $w^n = D^n + d = m$ and obtain the system of n equations :

(1.5)

$$\begin{cases} 1 = s_{0,1}\cdot t_{0,1} + m(s_{n-1,1}\cdot t_{1,1} + s_{n-2,1}\cdot t_{2,1} + \ldots + s_{1,1}\cdot t_{n-1,1}) \\ 0 = (s_{1,1}\cdot t_{0,1} + s_{0,1}\cdot t_{1,1}) + m(s_{n-1,1}\cdot t_{0,1} + s_{n-2,1}\cdot t_{2,1} + \ldots + s_{2,1}\cdot t_{n-1,1}) \\ 0 = (s_{2,1}\cdot t_{0,1} + s_{1,1}\cdot t_{1,1} + s_{0,1}\cdot t_{2,1}) + m(s_{n-1,1}\cdot t_{3,1} + s_{n-2,1}\cdot t_{4,1} + \ldots + s_{3,1}\cdot t_{n-1,1}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = (s_{n-2,1}\cdot t_{0,1} + s_{n-3,1}\cdot t_{1,1} + \ldots + s_{0,1}\cdot t_{n-2,1}) + m(s_{n-1,1}\cdot t_{n-1,1}) \\ 0 = (s_{n-1,1}\cdot t_{0,1} + s_{n-2,1}\cdot t_{1,1} + \ldots + s_{0,1}\cdot t_{n-1,1}) \end{cases}$$

Taking the $t_{i,1}$'s as unknows , the determinant of the system is $|A|$, where A is as in (1.1) .

Since $w^n = m$, $Q(w)$ has a basis $1, w, \ldots, w^{n-1}$, so that any algebraic number in $Q(w)$ has the form $\sigma = x_1 + x_2 w + \ldots + x_n w^{n-1}$, $x_i \in Q$ $(i = 1,\ldots,n-1)$. The norm of σ is a polynomial in the x_i and $N(\sigma) = \pm 1$ so we have $(-1)^{n-1} S_n = m$, $S_j = 0$ for $j = n$, $\sigma = e$ and S_n and S_j are the elementary symmetric functions in $D_0, D_1, \ldots, D_n$ and

$(-1)^n S_n = (-1)^n D_0 D_1 \cdots D_{n-1} = d$.

In this case

(1.6) $N(e) = \begin{vmatrix} s_{0,} \\ ms_{n-} \\ \cdots \\ ms_{2,} \\ ms_{1,} \end{vmatrix}$

which is the transpo

Also , $N(e) =$

$N(w-D) = \begin{vmatrix} -D & 1 \\ 0 & -i \\ \cdots & \cdots \\ 0 & 0 \\ m & 0 \end{vmatrix}$

and $N(e) = 1$.

Now we apply

(1.7) $t_{0,1} = \begin{vmatrix} 1 & ms_{n-1} \\ 0 & s_{0,1} \\ \cdots & \cdots \\ 0 & s_{n-3,} \\ 0 & s_{n-2,} \end{vmatrix}$

compared with (1.

$s_{0,1} = \frac{m + (-1)^n D}{d}$

$s_{i,1} = (-1)^{n-i}\binom{n}{n-i}$

In this case

$$(1.6)\quad N(e)=\begin{vmatrix} s_{0,1} & s_{1,1} & \cdots & s_{n-2,1} & s_{n-1,1} \\ ms_{n-1,1} & s_{0,1} & \cdots & s_{n-3,1} & s_{n-2,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ ms_{2,1} & ms_{3,1} & \cdots & s_{0,1} & s_{1,1} \\ ms_{1,1} & ms_{2,1} & \cdots & ms_{n-1,1} & s_{0,1} \end{vmatrix}$$

which is the transpose of the $|A|$ with A in (1.1).

Also, $N(e)=N\left[\frac{(w-D)^n}{d}\right]=\left[N\left(\frac{w-D}{d}\right)\right]^n$ and

$$N(w-D)=\begin{vmatrix} -D & 1 & 0 & \cdots & 0 & 0 \\ 0 & -D & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -D & 1 \\ m & 0 & 0 & \cdots & 0 & -D \end{vmatrix}=(-D)(-1)^{n-1}D^{n-1}+(-1)^{n-1}m=(-1)^{n-1}d$$

and $N(e)=1$.

Now we apply Cramer's rule to solve for $t_{0,1}$ in (1.5) and

$$(1.7)\ t_{0,1}=\begin{vmatrix} 1 & ms_{n-1,1} & \cdots & ms_{2,1} & ms_{1,1} \\ 0 & s_{0,1} & \cdots & ms_{3,1} & ms_{2,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & s_{n-3,1} & \cdots & s_{0,1} & ms_{n-1,1} \\ 0 & s_{n-2,1} & \cdots & s_{1,1} & s_{0,1} \end{vmatrix}$$ and $e=\frac{(w-D)^n}{d}$ if expanded and

compared with (1.4) gives (1.2)

$$\begin{cases} s_{0,1}=\dfrac{m+(-1)^n D^n}{d} \\ s_{i,1}=(-1)^{n-i}\dbinom{n}{n-i}\dfrac{D^{n-i}}{d} \end{cases}\qquad (i=1,\ldots,n-1)$$

(1.7) solves for $t_{0,1}$ in terms of the parametric forms for the (1.2).

Another way to write $t_{0,1}$ is :

$e^{-1} = d(w-D)^{-n} = \frac{d}{m}\left(1-\frac{D}{w}\right)^{-n}$ and since $\left|\frac{D}{w}\right| < 1$, we have

$$e^{-1} = \frac{d}{m}\sum_{j=0}^{\infty}\binom{n+j-1}{j}\left(\frac{D}{w}\right)^{j}.$$

If $(k-1)^n < j \le kn$, then $\frac{1}{w^j} = \frac{w^{kn-j}}{m^k}$ and $0 \le kn-j \le n-1$.

Let regroup the terms of our infinite series in the powers of w .

$$\text{(1.8)}\quad \begin{aligned} e^{-1} &= \frac{d}{m}\sum_{j=0}^{\infty}\binom{(j+1)n-1}{jn}\frac{D^{jn}}{m^j} + \left(\frac{d}{m}\sum_{j=0}^{\infty}\binom{jn+2n-2}{jn+n-1}\frac{D^{jn+n-1}}{m^{j+1}}\right)w + \\ &+ \cdots + \left(\frac{d}{m}\sum_{j=0}^{\infty}\binom{jn+n+1}{jn+2}\frac{D^{jn+2}}{m^{j+1}}\right)w^{n-2} + \left(\frac{d}{m}\sum_{j=0}^{\infty}\binom{jn+n}{jn+1}\frac{D^{jn+1}}{m^{j+1}}\right)w^{n-1}. \end{aligned}$$

From (1.8) we have

$$\text{(1.9)}\quad t_{0,1} = \frac{d}{m}\sum_{j=0}^{\infty}\binom{(j+1)n-1}{jn}\left(\frac{D^n}{m}\right)^{j}.$$

(1.9) gives a parametric family of infinite series and an explicit form of their sums as

$$\sum_{j=0}^{\infty}\binom{(j+1)n-1}{jn}\left(\frac{D^n}{m}\right)^{j} = \frac{m}{d}|A_1|$$

or $$\sum_{j=0}^{\infty}\binom{(j+1)n-1}{jn}\left(\frac{m}{D^n}\right)^{j} = \frac{m}{d}|A_1| \quad \text{as (1.3)}$$

where $|A_1|$ is $|A|$ with the first column replaced by the vector $(1,0,\ldots,0)$, and $s_{0,1}$ and $s_{i,1}$ as in (1.2).

Example for n=4. $\qquad m = w^4 = D^4 + d$

$$\sum_{j=0}^{\infty} \binom{4j+3}{4j} \left(\frac{D^4}{m}\right)^j = \frac{m}{d}|A_1|$$

$$|A_1| = \begin{vmatrix} 1 & ms_{3,1} & ms_{2,1} & ms_{1,1} \\ 0 & s_{0,1} & ms_{3,1} & ms_{2,1} \\ 0 & s_{1,1} & ms_{0,1} & ms_{3,1} \\ 0 & s_{2,1} & s_{1,1} & s_{0,1} \end{vmatrix}$$

$s_{0,1} = \dfrac{m + (-1)^4 D^4}{d}$ $\qquad$ and

$s_{i,1} = (-1)^{4-i} \binom{4}{3} \dfrac{D^{4-i}}{d}$ $\qquad$ i=1,2,3.

$$(1.10) \quad |A_1| = \begin{vmatrix} 1 & -4\frac{D}{d} & 6\frac{D^2}{d} & -4\frac{D^3}{d} \\ 0 & \frac{m+D^4}{d} & -4\frac{D}{d} & 6\frac{D^2}{d} \\ 0 & -4\frac{D^3}{d} & \frac{m+D^4}{d} & -4\frac{D}{d} \\ 0 & 6\frac{D^2}{d} & -4\frac{D^3}{d} & \frac{m+D^4}{d} \end{vmatrix}$$

$$ms_{3,1} = (-1)^{4-3} \binom{4}{1} \frac{D^{4-3}}{d} = -4\frac{D}{d}$$

$$ms_{2,1} = (-1)^{4-2} \binom{4}{2} \frac{D^{4-2}}{d} = 6\frac{D^2}{d}$$

$$ms_{1,1} = (-1)^{4-1} \binom{4}{3} \frac{D^{4-1}}{d} = -4\frac{D^3}{d}$$

$$ms_{0,1} = \frac{m + (-1)^4 D^4}{d} = \frac{m + D^4}{d} .$$

Solving (1.10) we get

$$\frac{m}{d}|A_1| = \frac{m}{d}\left[\left(\frac{m+D^4}{d}\right)^3 + \frac{28D^4m^2+28D^8m}{d^3}\right]$$

Let $n=4$, $D=4$, $d=2$ then $m=D^4+d=258$ and

$$\sum_{j=0}^{\infty}\binom{4j+3}{4j}\left(\frac{128}{129}\right)^j = 135794945 .$$

In conclusion (1.7) and (1.9) provide us with more classes of series and their sums since we can also solve for each of the $t_{i,1}$ $(i=1,\dots,n-1)$.

For example if $i=n-2$ we have

(1.11) $$t_{n-2,1} = \frac{dD^2}{m^2}\sum_{j=0}^{\infty}\binom{(j+1)n+1}{jn+2}\left(\frac{D^n}{m}\right)^j .$$

and

(1.12) $$t_{n-2,1} = \begin{vmatrix} s_{0,1} & ms_{n-1,1} & \cdots & 1 & ms_{1,1} \\ s_{1,1} & s_{0,1} & \cdots & 0 & ms_{2,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n-1,1} & s_{n-2,1} & \cdots & 0 & s_{0,1} \end{vmatrix}$$

(1.13)

where $s_{i,1}$ and $s_{0,1}$ are as in (1.2)

or $$\sum_{j=0}^{\infty}\binom{(j+1)n-i-1}{jn-i}\frac{D^{jn-1}}{m^j} = \left(\frac{m}{d}\right)|A_{i+1}|$$

as (1.3) where $s_{i,1}$ and $s_{0,1}$ are as in (1.2).

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