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SOME INFINITE SERIES AND SUMS FROM (BGEA)

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ABSTRACT. In this paper the author will use a tehnique to find sums of some infinite series from units in algebraic number fields. Starting with a unit $e = \frac{(w-D)^n}{d}$ from (BGEA) we compute e^k and e^{-k} powers of this unit in $Q(w)$, $w = \sqrt[n]{D^k + d}$, $D \in N$, $d \in Z$ and $d|D$ and we will evaluate the sums of some infinite series. In two previous papers [2], [3] the author used a similar method to find new combinatorial identities.

Key words and Phrases:	Abbreviation:
BAICA'S ALGORITHM IN A COMPLEX FIELD	(ACF)
JACOBI - PERRON ALGORITHM	(JPA)
(ACF) ALSO NAMED BAICA'S GENERALIZED EUCLIDEAN ALGORITHM	(BGEA)

1.Introduction. The (ACF) algorithm is an extension of the (JPA) over the complex numbers, and it is a very useful tool when it is periodic. The Euclidean Algorithm is a particular case of the (ACF) and we rightly can name (ACF) as Baica's Generalized Euclidean Algorithm (BGEA). In this paper we will use some previous results from the (BGEA) in the theory of units in the algebraic number fields [1].

We consider a unit $e = \frac{(w-D)^n}{d}$ from (BGEA) and use it to evaluate the sums of some infinite series.

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2.The statement of the problem

Theorem Let $e=(w-D)^n$ in $Q(w)$; $w=\sqrt[n]{D^n+d}$, $D\in N$, $d\in Z$, $d|D$, $k,n\in N$, $n>2$, $k>1$ and $m=D^n+d$.

$$(1.1)\quad A=\begin{pmatrix} s_{0,k} & ms_{n-1,k} & \cdots & ms_{2,k} & ms_{1,k} \\ s_{1,k} & s_{0,k} & \cdots & ms_{3,k} & ms_{2,k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n-2,k} & s_{n-3,k} & \cdots & s_{0,k} & ms_{n-1,k} \\ s_{n-1,k} & s_{n-2,k} & \cdots & s_{1,k} & s_{0,k} \end{pmatrix}$$

where

$$(1.2)\quad \begin{cases} s_{0,k}=\dfrac{1}{d^k}(m^k+(-1)^n\dbinom{kn}{n}D^n m^{k-1}+(-1)^{2n}\dbinom{kn}{2n}D^{2n}m^{k-2}+ \\ \quad +\ldots+(-1)^{kn}\dbinom{kn}{kn}D^{kn}) \\ s_{i,k}=\dfrac{1}{d^k}((-1)^{n-i}\dbinom{kn}{n-i}D^{n-i}m^{k-1}+(-1)^{2n-i}\dbinom{kn}{2n-i}D^{2n-i}m^{k-2}+ \\ \quad +\ldots+(-1)^{kn-i}\dbinom{kn}{kn-i}D^{kn-i}) \qquad i=1,\ldots,n-1 \end{cases}$$

Then

$$(1.3)\quad \begin{cases} \displaystyle\sum_{j=0}^{\infty}\binom{(j+k)n-1}{jn}\frac{D^{jn}}{m^j}=\left(\frac{m}{d}\right)^k|A_1| \\ \displaystyle\sum_{j=1}^{\infty}\binom{(j+k)n-i-1}{jn-i}\frac{D^{jn-i}}{m^j}=\left(\frac{m}{d}\right)^k|A_{i+1}| \qquad i=1,\ldots,n-1 \end{cases}$$

where $|A_1|$ is $|A|$ with the first column replaced by the vector $(1,0,\ldots,0)$ and $|A_{i+1}|$ is $|A|$ with the i-th column replaced by the vector $(1,0,\ldots,0)$.

Proof

Suppose

$$(1.4)\qquad \begin{cases} e^k = s_{0,k} + s_{1,k}w + \cdots + s_{n-1,k}w^{n-1} \quad (k = 1,2,\ldots) \\ e^{-k} = t_{0,k} + t_{1,k}w + \cdots + t_{n-1,k}w^{n-1} \end{cases}$$

We perform $1 = e^k e^{-k}$ reducing the powers of w, knowing that $w^n = D^n + d = m$ and obtain the system of n equations :

$$(1.5)\qquad \begin{cases} 1 = s_{0,k}\cdot t_{0,k} + m(s_{n-1,k}\cdot t_{1,k} + s_{n-2,k}\cdot t_{2,k} + \\ \quad + \ldots + s_{1,k}\cdot t_{n-1,k}) \\ 0 = (s_{1,k}\cdot t_{0,k} + s_{0,k}\cdot t_{1,k}) + m(s_{n-1,k}\cdot t_{0,k} + \\ \quad + s_{n-2,k}\cdot t_{2,k} + \ldots + s_{2,k}\cdot t_{n-1,k}) \\ 0 = (s_{2,k}\cdot t_{0,k} + s_{1,k}\cdot t_{1,k} + s_{0,k}\cdot t_{2,k}) + \\ \quad + m(s_{n-1,k}\cdot t_{3,k} + s_{n-2,k}\cdot t_{4,k} + \ldots + s_{3,k}\cdot t_{n-1,k}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = (s_{n-2,k}\cdot t_{0,k} + s_{n-3,k}\cdot t_{1,k} + \ldots + s_{0,k}\cdot t_{n-2,k}) + \\ \quad + m(s_{n-1,k}\cdot t_{n-1,k}) \\ 0 = s_{n-1,k}\cdot t_{0,k} + s_{n-2,k}\cdot t_{1,k} + \ldots + s_{0,k}\cdot t_{n-1,k} \end{cases}$$

Taking the $t_{i,k}$'s as unknows , the determinant of the system is $|A|$, where A is as in (1.1) .

Since $w^n = m$, $Q(w)$ has a basis $1, w, \ldots, w^{n-1}$, so that any algebraic number in $Q(w)$ has the form $\sigma = x_1 + x_2 w + \ldots + x_n w^{n-1}$, $x_i \in Q$ $(i = 1,\ldots,n-1)$. The norm of σ is a polynomial in the x_i and $N(\sigma) = \pm 1$. So we have $(-1)^{n-1} S_n = m$, $S_j = 0$ for $j = n$, $\sigma = e$ and S_n and S_j are the elementary symmetric functions in $D_0, D_1, \ldots, D_n$ and

$$(-1)^n S_n = (-1)^n D_0 D_1 \cdots D_{n-1} = d\ .$$

In this case

$$(1.6)\quad N(e^k)=\begin{pmatrix} s_{0,k} & s_{1,k} & \cdots & s_{n-2,k} & s_{n-1,k} \\ ms_{n-1,k} & s_{0,k} & \cdots & s_{n-3,k} & s_{n-2,k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ ms_{2,k} & ms_{3,k} & \cdots & s_{0,k} & s_{1,k} \\ ms_{1,k} & ms_{2,k} & \cdots & ms_{n-1,k} & s_{0,k} \end{pmatrix}$$

which is the transpose of the $|A|$ with A in (1.1).

Also, $N(e^k)=N\left[\dfrac{(w-D)^{kn}}{d^k}\right]=\left[N\left(\dfrac{w-D^k}{d^k}\right)\right]^n$

and

$$N(w-D)=\begin{vmatrix} -D & 1 & 0 & \cdots & 0 & 0 \\ 0 & -D & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -D & 1 \\ m & 0 & 0 & \cdots & 0 & -D \end{vmatrix}=(-D)(-1)^{n-1}D^{n-1}+(-1)^{n-1}m=(-1)^{n-1}d$$

and $N(w-D)=N(e)=1$.

Since $N(w-D)=1$. Then $N(w-D)^k=1$.

Now we apply Cramer's rule to solve for $t_{0,k}$ in (1.5) and

$$(1.7)\qquad t_{0,k}=\begin{pmatrix} 1 & ms_{n-1,k} & \cdots & ms_{2,k} & ms_{1,k} \\ 0 & s_{0,k} & \cdots & ms_{3,k} & ms_{2,k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & s_{n-3,k} & \cdots & s_{0,k} & ms_{n-1,k} \\ 0 & s_{n-2,k} & \cdots & s_{1,k} & s_{0,k} \end{pmatrix}$$

and

$$e^k = \frac{(w-D)^{kn}}{d^k} = \frac{1}{d^k}\left(w^{kn} - \binom{kn}{1} w^{kn-1} D + \ldots + (-1)^n \binom{kn}{n} w^{kn-n} D^n + \ldots \right.$$
$$\left. + (-1)^{2n} \binom{kn}{2n} w^{kn-2n} D^{2n} + \ldots + (-1)^{kn} \binom{kn}{kn} D^{kn} \right).$$

and since $w^n = m$,

$$s_{0,k} = \frac{1}{d^k}\left(m^k - (-1)^n \binom{kn}{n} D^n m^{k-1} + (-1)^{2n} \binom{kn}{2n} D^{2n} w^{k-2} + \ldots + (-1)^{kn} \binom{kn}{kn} D^{kn} \right)$$

$$s_{i,k} = \frac{1}{d^k}\left((-1)^{n-i} \binom{kn}{n-i} D^{n-i} m^{k-1} + (-1)^{2n-i} \binom{kn}{2n-i} D^{2n-i} w^{k-2} + \ldots + (-1)^{kn-i} \binom{kn}{kn-i} D^{kn-i} \right)$$
$$i = 1, \ldots, n-1.$$

(1.7) solves for $t_{0,k}$ in terms of the parametric forms for the (1.2). Another way to write $t_{0,k}$ is $e^{-k} = d^k (w-D)^{-nk} = \left(\frac{d}{m}\right)^k \left(1 - \frac{D}{w}\right)^{-nk}$ and since $\left|\frac{D}{w}\right| < 1$ we have

$$e^{-k} = \left(\frac{d}{m}\right)^k \sum_{j=0}^{\infty} \binom{jn+kn-1}{jn} \frac{D^{jn}}{m^j} + \left(\frac{d^k}{m^k} \sum_{j=0}^{\infty} \binom{jn+(k+1)n-2}{jn+n-1} \frac{D^{jn+n-1}}{m^{j+1}} \right) w +$$
(1.8)
$$+ \cdots + \left(\frac{d^k}{m^k} \sum_{j=0}^{\infty} \binom{jn+kn+1}{jn+2} \frac{D^{jn+2}}{m^{j+1}} \right) w^{n-2} + \left(\frac{d^k}{m^k} \sum_{j=0}^{\infty} \binom{jn+kn}{jn+1} \frac{D^{jn+1}}{m^{j+1}} \right) w^{n-1}.$$

In particular from (1.8) we have

$$t_{0,k} = \left(\frac{d}{m}\right)^k \sum_{j=0}^{\infty} \binom{jn+kn-1}{jn} \left(\frac{D^n}{m}\right)^j$$

or

(1.9) $$\begin{cases} t_{0,k} = \left(\frac{d}{m}\right)^k \sum_{j=0}^{\infty} \binom{(j+k)n-1}{jn} \left(\frac{D^n}{m}\right)^j \\ t_{i,k} = \left(\frac{d}{m}\right)^k \sum_{j=0}^{\infty} \binom{(j+k)n-i-1}{jn-i} \frac{D^{jn-1}}{m^j} \end{cases}$$

$i = 1,\dots,n-1.$

(1.9) gives a parametric family of infinite series and an explicit form of their sums is given by (1.7) as (1.3) with $s_{0,k}$ and $s_{i,k}$ as in (1.2).

Example for n=5 and k=1, $m = w^4 = D^4 + d$ and $d|D$.

(1.10) $$\frac{m}{d}|A_1| = \sum_{j=0}^{\infty} \binom{5j+4}{5j} \left(\frac{D^5}{m}\right)^j = \left(\frac{m}{d}\right)\left(1 + \frac{375D^5m}{d^2} + \right.$$
$$\left. + \frac{1500D^{10}m - 1500D^5m^2}{d^3} + \frac{1250D^{15}m - 1875D^{10}m^2 + 1250D^5m^3}{d^4}\right)$$

Since $|A_1| = \begin{vmatrix} 1 & ms_{4,1} & ms_{3,1} & ms_{2,1} & ms_{1,1} \\ 0 & s_{0,1} & ms_{4,1} & ms_{3,1} & ms_{2,1} \\ 0 & s_{1,1} & s_{0,1} & ms_{4,1} & ms_{3,1} \\ 0 & s_{2,1} & s_{1,1} & s_{0,1} & ms_{4,1} \\ 0 & s_{3,1} & s_{2,1} & s_{1,1} & s_{0,1} \end{vmatrix}$

$$s_{0,1} = \frac{m + (-1)^5 D^5}{d}$$ and

$$s_{i,1} = (-1)^{5-i}\binom{5}{5-i}\frac{D^{5-i}}{d} \qquad \text{i=1,2,3,4}$$

$$s_{0,1} = \frac{m - D^5}{d}\ ; \quad s_{2,1} = -10\frac{D^3}{d}\ ; \quad s_{4,1} = -5\frac{D}{d}\ ;$$

$$s_{1,1} = 5\frac{D^4}{d}\ ; \quad s_{3,1} = 10\frac{D^2}{d}$$

and

$$|A_1| = \begin{vmatrix} 1 & -5m\frac{D}{d} & 10m\frac{D^2}{d} & -10m\frac{D^3}{d} & 5m\frac{D^4}{d} \\ 0 & \frac{m-D^5}{d} & -5m\frac{D}{d} & 10m\frac{D^2}{d} & -10m\frac{D^3}{d} \\ 0 & 5\frac{D^4}{d} & \frac{m-D^5}{d} & -4\frac{D}{d} & 10m\frac{D^2}{d} \\ 0 & -10\frac{D^3}{d} & 5\frac{D^4}{d} & \frac{m-D^5}{d} & -5m\frac{D}{d} \\ 0 & 10\frac{D^2}{d} & -10\frac{D^3}{d} & 5\frac{D^4}{d} & \frac{m-D^5}{d} \end{vmatrix}$$

or

$$|A_1| = 1 + \frac{375D^5 m}{d^2} + \frac{1500D^{10}m - 1500D^5 m^2}{d^3} + \\ + \frac{1250D^{15}m - 1875D^{10}m^2 + 1250D^5 m^3}{d^4}$$

which implies (1.10).

For numerical $d, D \in N$ and $d|D$ we will get a number for the sum in (1.10).

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