AN ABRIDGED METHOD TO DERIVE THE ASYMPTOTIC FORMULA FOR THE GOLDBACH DECOMPOSITIONS

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1. Introduction

In ref.[1] the authors have given a method in order to obtain an exact formula for the Hardy-Littlewood function.

$$v(t) = \sum_{t=p_1+p_2} \log p_1 \,.\, \log p_2$$

Here is indicated how the use of the tauberian theorem quoted in Lemma 3 of ref.[1] enables us to obtain a much shorter derivation of the asymptotic formula for v(t).

2. The starting formula

As was proved in ref.[1], we have that

$$L(v(t)) = e^s\left(\frac{1-e^{-s}}{s}\right)^2 \left\{\sum_q^N \sum_h \frac{\mu(q)}{\varphi(q)(s+2\pi i\, h/q)} + AN^{2\vartheta+1/2+\varepsilon}\right\}^2$$

where L denotes the Laplace transform, ϑ is the upper bound of the real part of the imaginary zeros of the L-series involved, and the formula is valid of $\vartheta \geq ¾$, which is the actual case.

We write it as

$$L\{v(t)\} = e^s\left(\frac{1-e^{-s}}{s}\right)^2 \left\{g_N(s) + AN^{2\vartheta+1/2+\varepsilon}\right\}^2$$

$$= e^s\left(\frac{1-e^{-s}}{s}\right)^2 \left\{g_N{}^2(s) + 2g_N(s)AN^{2\vartheta+1/2+\varepsilon} + A^2N^{4\vartheta+1+2\varepsilon}\right\}$$

(2.1)

Due to the fact that

$$L^{-1}\left\{e^s\left(\frac{1-e^{-s}}{s}\right)^2\right\} = \begin{cases} t & \text{if } 0<t<1 \\ 2-t & \text{if } 1<t<2 \\ 0 & \text{otherwise} \end{cases}$$

the term at the right hand side of (2.1) has not any relevance for v(t), and we can put

$$L\{v(t)\} = e^s\left(\frac{1-e^{-s}}{s}\right)^2 \left\{g_N{}^2(s) + 2g_N(s)AN^{2\vartheta+1/2+\varepsilon}\right\} \qquad \text{if } t>2.$$

By the tauberian theorem of Lemma 3 of ref.[1]

$$v(t) \sim L^{-1}\left\{g_N{}^2(s) + 2AN^{2\vartheta+1/2+\varepsilon}\, g_N(s)\right\} \tag{2.1}$$

$$= L^{-1}\left\{\sum\sum \frac{\mu^2(q)}{\varphi^2(q)(s+2\pi i\, h/q)^2} + \sum \sum_{q_1 \neq q_2} \sum \sum \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\,\varphi(q_2)(s+2\pi i\, h_1/q_1)(s+2\pi i\, h_2/q_2)}\right.$$

$$\left. + 2AN^{2\vartheta+1/2+\varepsilon} \sum\sum \frac{\mu(q)}{\varphi(q)(s+2\pi i\, h/q)}\right\}$$

$$= \sum_{q=1}^{N} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t)\,.\,t + \sum\sum\sum\sum \frac{\mu(q_1)\ \mu(q_2)}{\varphi(q_1)\ \varphi(q_2)} \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1} +$$

$$+ 2AN^{2\vartheta+1/2+\varepsilon} \sum_{q=1}^{N} \frac{\mu(q)}{\varphi(q)} C_q(t)$$

But

$$\sum_{q=1}^{N} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t)\,.\,t \quad = \quad \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t)\,.\,t \ + \ \delta_1\, e^{3\gamma} d(t) \frac{(\log\ \log N)^2}{N} \log\log t\,.\,t$$

$$\left|\sum\sum\sum\sum \frac{\mu(q_1)\,\mu(q_2)}{\varphi(q_1)\,\varphi(q_2)} \frac{e^{-A_1 t} - e^{-A_2 t}}{A_2 - A_1}\right| \leq \frac{\delta_2}{2\pi} N^2 (N+1)^2 \tag{2.2}$$

$$\vartheta(t) - \vartheta(t-1) \sim \sum_{q=1}^{N} \frac{\mu(q)}{\varphi(q)} C_q(t)$$

(Where $\vartheta(t)$ is the Chebishev function $\vartheta(t) = \sum_{p \leq t} \log p$)
as was shown in ref.[1] by Lemmas 1,2 and 3.

Hence

$$v(t) \sim \sum_{1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t) \cdot t + \delta_1\, t\, e^{3\gamma}\ d(t) \frac{(\log\log N)^2}{N} \log\log t$$

$$+ \frac{\delta_2}{2\pi} N^2 (N+1)^2 + \delta_3\, 2AN^{2\vartheta+1/2+\varepsilon} \log t \tag{2.3}$$

We choose now $t = N^5$, so that

$$v(t) \sim \sum_{1}^{\infty} \frac{\mu^2(q)}{\varphi^2(q)} C_q(t) \cdot t + \delta_1\, e^{3\gamma}\ d(t)(\log\log t)^3 . \, t^{4/5}$$

$$+ \frac{\delta_2}{2\pi} t^{4/5} + \delta_3\ 2\,A\, t^{\frac{2}{5}\vartheta + \frac{1}{10} + \varepsilon} \log t \tag{2.4}$$

It is evident now the little influence that the value of ϑ has upon the value of v(t).
The preceding formula coincides, in its essential features, with that deduced by the exact method.
Due to the multiplicative properties of $\mu(q)$, $\varphi(q)$ and $C_q(t)$ the series in the first term at right can be written as:

$$\sum_{1}^{\infty}\frac{\mu^2(q)}{\varphi^2(q)}C_q(t)=2\prod_{p=3}^{\infty}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|t}\frac{p-1}{p-2}$$

$$=1{,}3203\prod_{p|t}\frac{p-1}{p-2} \qquad (2.5)$$

So that (2.4) turns out to be

$$v(t)\sim 1{,}3203\prod_{p|t}\frac{p-1}{p-2}.t+0(t^{\frac{4}{5}+\varepsilon})$$

From (2.4) follows, as was shown in ref.[1], that the Goldbach hypothesis is correct for even $t>10^{60}$.

REFERENCE

[1] M. Baica – A. Peretti._ The binary Goldbach problem – www.peretti.da.ru