

**THE PARATRIGONOMETRY AND SOME
OF ITS SPECIFIC SYMMETRIES**

Malvina BAICA and Mircea CÂRDU

Abstract. In this paper the authors provide a systematisation of all nonconventional trigonometries, other than the Classical Trigonometry (CT) [7], considering that these are: Quadratic Trigonometry (QT) [1], Polygonal Trigonometry (PT) [6], Transtrigonometry (TT) [3], Infratrigonometry (IT) [4] and Ultratrigonometry (UT) [5]. We adopt this generic denomination of Paratrigonometry (PRT) which includes these mentioned nonconventional trigonometries, as well as the Classical (conventional) Trigonometry (CT).

Also, we will analyse some symmetric situations for the functions which represent mathematical modelling of the basic trigonometric figures of Paratrigonometry (as well as "Trigonometric circle" of CT, for example).

Keywords and phrases: Nonconventional trigonometries, Paratrigonometry, Symmetric functions.

1. Introduction

In some of the authors previous papers, [3], [4], and [5], there were analysed the "trigonometric" functions which are based on two fundamental relations common to all types of introduced trigonometries as such:

- Transtrigonometry (TT) [3];
- Infratrigonometry (IT) [4];
- Ultratrigonometry (UT) [5].

These two fundamental relations could be written, in a general form, as:

$$|\operatorname{spr}_k \alpha|^k + |\operatorname{cpr}_k \alpha|^k = 1 \tag{1.1}$$

$$\operatorname{tpra} = \operatorname{tga} \tag{1.2}$$

where $0 \leq k \leq \infty$.

The index k define the "order" of the function which we will use in the sequel [3]. To tpra function we did not attach any index k , since the value of this function does not depend on the value of k .

For TT and UT cases ($1 < k \leq \infty$), BTFs form is of some rhombuses with curved sides and concavity to the direction of the reference point O, thus in an opposite sense to the concavity of the rhombuses with curved sides in IT. At the limit, when $k = \infty$, the curvature of these sides becomes extreme, they fragments in the points A, B, C, and D. In Figure 1, in this way, BTF for $k = \infty$ becomes the square ABCD.

In consequence, it is to notice the fact that, from the BTFs point of view, the form of these figures can be part of two categories, namely: first, the one characteristic to IT and the second, characteristic to TT and UT. The limit between these two categories is QT ($k = 1$) where BTF is the rhombus with straight sides (without concavity) QMNP.

In the second category, a special case is the characteristic "trigonometric circle" in CT ($k = 2$).

Finally, we can say that, from the point of view of the graphical representation of $\text{spr}_k \alpha$ and $\text{cpr}_k \alpha$ functions, the border between these two distinct zones is the trigonometric circle, and from the point of view of BTFs, the border between the two distinct zones is the trigonometric rhombus with the straight lines in QT. The zone between these two borders characterized by $1 < k < 2$, we named it as Transtrigonometry (TT) [3].

For our analysis which we carry on, because of some reasons which we will explain in detail in the following chapter, we will have in our view as a guide the trigonometric rhombus with straight sides (BTF in QT). This divides the entire domain $0 \leq k \leq \infty$ in two areas, namely: IT ($0 \leq k < 1$) and respectively TT ($1 < k < 2$) together with UT ($2 < k \leq \infty$), including CT ($k = 2$). We named these last two trigonometries (TT and UT) together with the "Extratrigonometry" (ET) which, regarding to Figure 1, takes possession of the entire zone exterior to the rhombus QMNP ($k = 1$).

Regarding Table 1, we have to mention that the reasons for which we name the trigonometric rhombus in CT as the "Paratrigonometric mirror" will be explained in the next chapter. In this chapter we will analyse some noticed particularities of symmetries regarding BTFs of PRT.

3. The basic trigonometric figures (BTFs) of Paratrigonometry and their implicit characteristic equations

In Figure 1, we represented the most important BTFs of PRT, namely: the Cross ($k = 0$), the Trigonometric Rhombus with straight sides MNPQ ($k = 1$), the Trigonometric Circle ($k = 2$) and the Trigonometric Square ABCD ($k = \infty$). Also, there are represented some of PTFs of IT ($k = 0.4$ and $k = 0.8$), TT ($k = 1.4$) and UT ($k = 8$).

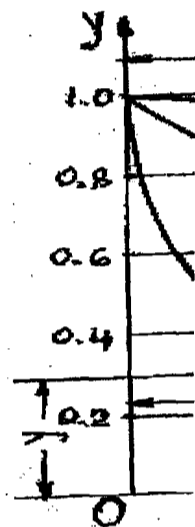
From their analysis we observe two symmetries, namely:

- The curve symmetry in relation with the two coordinate axis for a specific value of k , which is found in one of the trigonometric quadrants;
- The symmetry of a such curve situated "above" to the rhombus side QM (if we refer to the first quadrant) with a corresponding curve from the space "bellow" the respective side. In another way saying, BTFs from ET have symmetries in IT and

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conversely. The line QM is its own symmetry and all other symmetries are related to this one like being a mirror. For this reason we named BTF for QT (k = 1) "The Rhombus with paratrigonometric mirrors" (see Table 1), having in our view the fact that everything what we have said before regarding to the first quadrant are valid for the other (II - IV) quadrants, also.

In order to analyse from the mathematical point of view these symmetries, we apply to the formula which represent the side in the first quadrant - for simplicity - of BTF.

This formula was represented and explained in [3] and used in [4] and [5] and it is valid for the entire domain $0 \leq k \leq \infty$ and thus it represents a fundamental formula in PRT.

It express the relation between the coordinates x and y of the system Ox - Oy in which BTF is represented and has the form:

$$y = (1 - |x|^k)^{1/k} \tag{3.1}$$

In Figure 2 there are represented the sides from the first quadrant of BTFs for $k = 1, 1 < k < \infty$ (of ET domain) and $0 < \kappa < 1$ (of IT domain). We denoted by the Greek symbol κ the order value of the function from IT (to distinct from k of ET) for a reason which we will soon speak about it.

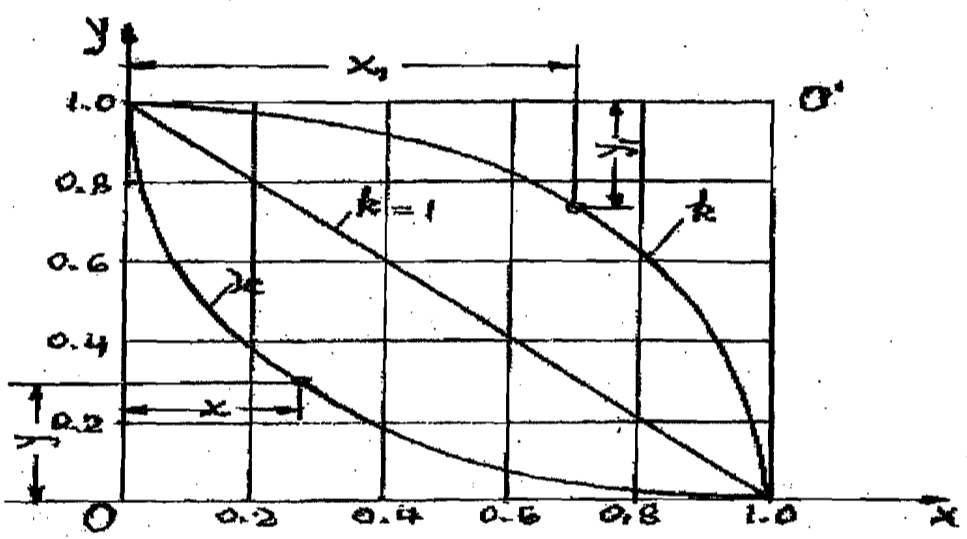


Fig.2

In order to prove the symmetry of a) above it is sufficient to set formula (3.1) under the form:

$$x = (1 - |y|^k)^{1/k} \tag{3.2}$$

The symmetry of formulas (3.1) and (3.2) direct us to conclude that there exists BTFs symmetry in relation with Ox and Oy coordinate axis.

Since we only refer to the first quadrant, as we have mentioned before, in formulas (3.1) and (3.2) it is no more in need to use "absolute values" for x and y

(always positive in this quadrant) and the corresponding formulas can be written under a simpler form:

$$y = (1-x^k)^{1/k} \quad (3.3)$$

$$x = (1-y^k)^{1/k} \quad (3.4)$$

In what follows, we will use formula (3.4) applied in ET, as we have mentioned before. On the other side, for IT this formula will have the form:

$$y = (1-x^\kappa)^{1/\kappa} \quad (3.5)$$

To analyse the symmetry type mentioned at the point b) above, we consider that the curves characterized by k and κ of Figure 2 are symmetric with respect to the line $k = 1$. In another way said, the curve κ is the "mirror" image (with respect to $k = 1$) of curve k , and conversely.

We denote the present coordinate of curve κ with x and y , and the present values of curve k with x_1 and y_1 . We apply formulas (3.3) and (3.5) for the curves k and κ case. For curve k case we have:

$$y_1 = (1-x_1^k)^{1/k} \quad (3.6)$$

and for curve κ case, formula (3.5) is also valid.

Referring to Figure 2, we can write $x_1 = (1-x)$ and $y_1 = (1-y)$. Introducing x_1 and y_1 in formula (3.6) in this way expressed as functions of x and y and using formula (3.5) to express y as function of x , we obtain the relation:

$$1 - [1 - (1-x)^k]^{1/k} = (1-x^\kappa)^{1/\kappa} \quad (3.7)$$

If we take logarithm of relation (3.7) we obtain the following banding formula between κ and k :

$$\kappa = [\ln(1-x^\kappa)] / A \quad (3.8)$$

where

$$A = \ln\{1 - [1 - (1-x)^k]^{1/k}\} \quad (3.9)$$

We see that from formula (3.8) we can not find κ explicitly. In other words, formula (3.8) is implicitly given regarding κ (and also k).

In any case, the values of κ and k characterise two symmetric curves in ET and respectively in IT given the "mirror" having $k = 1$. If we symbolise the symmetry "status" by " Σ_{im} ", we can write $\Sigma_{im} k \rightarrow \kappa$. Evidently, we can also write $\Sigma_{im} \kappa \rightarrow k$, or simply $\Sigma_{im} k \leftrightarrow \Sigma_{im} \kappa$.

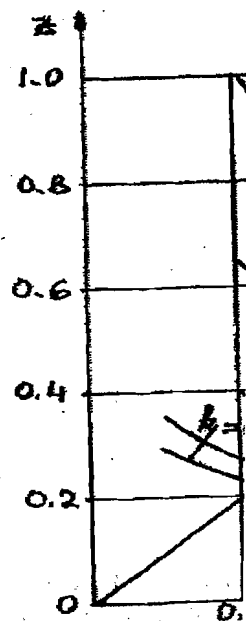
Since κ and k respectively can be find in an implicit form in the formula (3.8), to determine κ as a function of k we proceed graphically as we will continue to show.

On the left side of the equality sign in formula (3.8) we replace κ by z . In this way we will obtain the following formula which express z as a function of k :

$$z = [\ln(1-x^z)] / A \quad (3.10)$$

where A is given by formula (3.9).

In this way $z = z(\kappa)$ have both parameters k and κ . Giving various values for k (in the domain $1 < k < \infty$) and for x (in the domain $0 < x < 1$) we obtain various curves representing $z = z(\kappa)$, as it can be seen in the Figure 3.



If for a certain $z(\kappa)$ intersect in a point, namely $z = \kappa$. In Figure 3 curves $z(\kappa)$ for $k = 2$ and $\kappa = 0.56$. Thus we can

$$(3.11)$$

Any other curve will pass through the point (z, z) in Figure 2 is symmetric with respect to the line $k = 1$. Evidently, the coordinate of this point is even: $z = \kappa$.

In Figure 3 the bisector of the right angle of this line is even: $z = \kappa$. values of k (see Figure 3).

4. The semicircle Paratrigon

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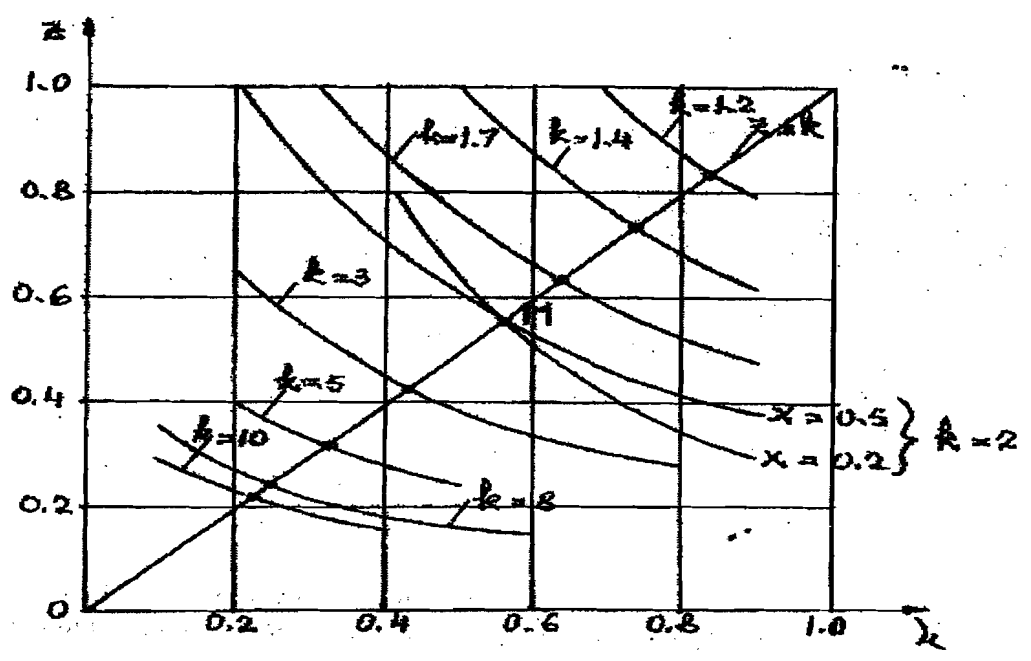


Fig. 3

If for a certain value of k we live two values for x , the curves of the function $z(\kappa)$ intersect in a point whose coordinates represent the solution of the problem, namely $z = \kappa$. In Figure 3, M is a such point which represent the intersection of the curves $z(\kappa)$ for $k = 2$ and for the values of $x = 0.5$ and $x = 0.2$ respectively. The result is $\kappa = 0.56$. Thus we can write:

$$\Sigma \text{im } (k = 2) \leftrightarrow \Sigma \text{im } (\kappa = 0.56)$$

(3.11)

Any other curve of the function $z(\kappa)$ for $k = 2$ corresponding other values of x will pass through the point M . This is because for any value of x the curve for κ of Figure 2 is symmetric with respect to the "mirror" $k = 1$, of the curve characterised by k . Evidently, the converse is valid also, as it is shown by formula (3.11).

In Figure 3 we see that the point M is on the line OM which represents the bisector of the right angle formed by the coordinate axis $Ox - Oy$. In fact the equation of this line is even $z = z(\kappa)$. The intersection of this line with other curves, for various values of k (see Figure 3), will give the solutions for $\Sigma \text{im } k \rightarrow \kappa$.

4. The semiempirical equation for the basic symmetric figures from Paratrigonometry

With values for $\kappa = \Sigma \text{im } k$, from Figure 3, we construct the curve $\kappa = \phi(k)$ represented in Figure 4.

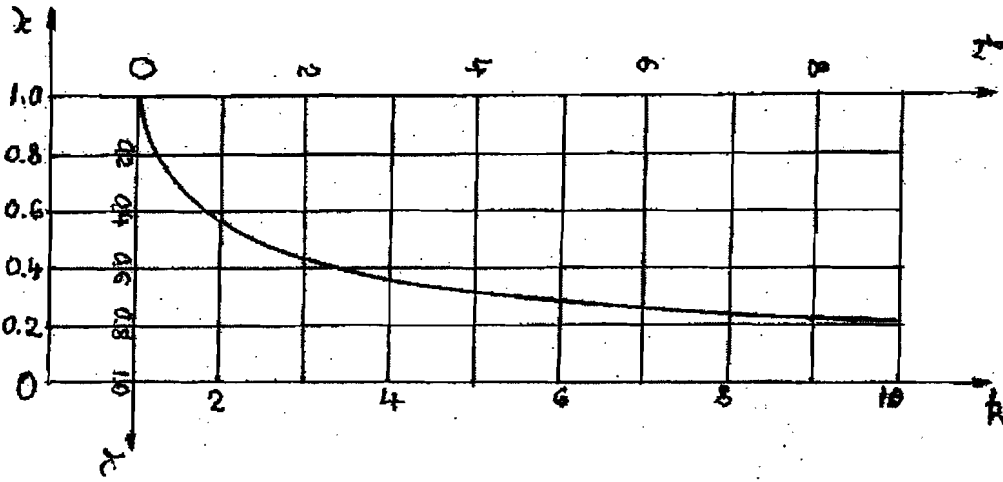


Fig.4

We see that, if considering another system of coordinates than the system with axis $Ok - O\kappa$, namely $Ok' - O\kappa'$, the curve mentioned before will have a form of the type representing exponential functions of the form:

$$k' = C \cdot e^{a \cdot \kappa'} \quad (4.1)$$

In order to have this function (4.1) expressed in the κ and k coordinates we replace κ' and k' by $\kappa' = (1 - \kappa)$ and $k' = (1 - k)$.

After making these substitutions, we apply again logarithm to the relation (4.1) and going the necessary operations for simplification we arrive to the following formula:

$$\kappa = \varepsilon - [\ln(k-1)] / a \quad (4.2)$$

By trying some values for constants ε and a , and comparing the resulting values from formula (4.2) with those of Figure 4, we obtain $\varepsilon = 0.56$ and $a = 6$. With these values, formula (4.2) becomes:

$$\kappa = 0.56 - [\ln(k-1)] / 6 \quad (4.3)$$

This formula (4.3) represents, with a high precision degree, this function $\kappa = \varphi(k)$ for the values of k in the domain $1.075 < k < 10$.

Formula (4.3) is not applicable for $k = 1$, but returning to formula (3.10) we conclude that for $k = 1$ introduced in formula (3.8) we have $z = \kappa$ only if $\kappa = 1$. That is

$$\Sigma \text{im}(k=1) \leftrightarrow \Sigma \text{im}(\kappa=1)$$

This last relation represents the mathematical expression of the "mirror" in the Paratrigonometry.

For the extreme cases marked by values $k = 0$ and respectively $k = \infty$ we apply the results of [4] and [5] mentioned before. Thus, in [4] we proved that, in IT, for $k = 0$ (now denoted $\kappa = 0$), the BTF is the Cross (OQ - OM - ON - OP of Figure 1), and in [5] we proved that, in UT, for $k = \infty$, the BTF is the square ABCD of Figure 1.

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5. Conclusion

5.1. All the [3], [4], [5] and [6] to Trigonometry (CT) developed in chapter

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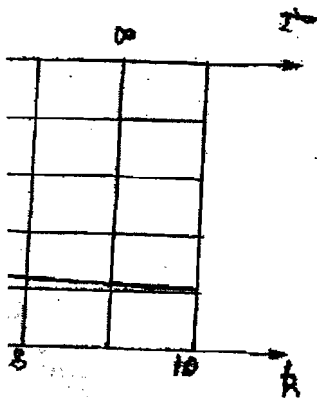
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1. The Basic (PT) dom
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From the geometrical point of view it can be seen in Figure 1 that the Cross $OQ - OM - ON - OP$ is symmetric to the square $ABCD$, in relation with the paratrigonometric rhombus of the mirrors. The same thing we can see in Figure 2, referring to the first quadrant. Using the above symbols, this thing can be expressed as:

$$\Sigma_{im}(k = \infty) \leftrightarrow \Sigma_{im}(\kappa = 0)$$

Again from the analysis of formula (4.3) we can see that for $k = 2$ we obtain $\kappa = 0.56$ as we shown in the previous chapter and we have found it using relation (3.11).

5. Conclusions

5.1. All the "Trigonometries" which we analysed in the previous papers [2], [3],[4],[5] and [6] together with the Quadratic Trigonometry (QT) [1] and the Classical Trigonometry (CT) [7] can be comprised in the notion of the Paratrigonometry (PRT) developed in chapter 2 of this paper.

The PRT structure and its relation with all the others trigonometries is given in Table 1. The basic relations from PRT, (1.1) and (1.2) can be applied in the case of all mentioned trigonometries, distinguishing themselves by the values for the "order" k .

This is also mentioned in Table 1, where we point out another classification elements, namely the Basic Trigonometric Figures (BTFs), as could be the Trigonometric circle in CT, Trigonometric rhombus with straight sides in QT etc.

5.2. Regarding BTFs, in PRT we established some BTFs symmetries, such these among BTFs from the Extratrigonometry (ET) with $1 < k \leq \infty$ and BTFs from the Infratrigonometry (IT) with $0 \leq k < 1$ in relation with the "Paratrigonometric mirror" of QT with $k = 1$ amply analysed in chapters 3 and 4.

Thus, in chapter 3 we established a semiempiric explicite equation for the respective symmetries. For a better expression of these symmetries, in IT case the order of the trigonometric functions was denoted by κ and the k notation was reserved for ET. We introduced the symbol " Σ_{im} ", and the symmetry between a BTF from ET (of order k) and a corresponding BTF of that from ET of order κ from IT, was denoted by $\Sigma_{im} k \leftrightarrow \Sigma_{im} \kappa$, the sign \leftrightarrow indicating the respective relation reciprocally.

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2. Symmetric BTFs given the mirror $k = 1$, represented in the first quadrant.
3. The representation of the function $z(\kappa)$ for $x = 0.5$ (in the case $k = 2$, also for $x = 0.2$) having as parameter distinct values for k .
4. The representation of function $\kappa = \varphi(k)$.

Table

1. The Paratrigonometry (PRT) structure

The name of PRT structure	The value domain of the order k	BTF form	The category as part of PRT	The generic denomination
Infratrigonometry (IT)	$0 \leq k < 1$	Rhombus with curves sides and the concavity opposite to the reference O^*	Infratrigonometry (IT)	Paratrigonometry (PRT) $0 \leq k \leq \infty$
Quadratic Trigonometry (QT)	$k = 1$	Rhombus with straight sides (Rhombus with "Paratrigonometric mirrors")	Quadratic Trigonometry (QT)	
Transtigonometry (TT)**	$1 < k < 2$	Rhombus with curves sides and the concavity towards the reference O^*	Extratrigonometry (ET) $1 < k \leq \infty$	
Classical trigonometry (CT)	$k = 2$	Trigonometric circle		
Ultratrigonometry (UT)	$2 < k \leq \infty$	Rhombuses with curved sides and the concavity towards the reference O^*		

- * The coordinate system $Ox - Oy$ origine
- ** Polygonal Trigonometry (PT) is referred to the domain $1 < k < 2$, but k is variable in function of the sides number in the Trigonometric Polygon

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