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BAICA & CÂRDU PARATRIGONOMETRIC FUNCTIONS RAISED TO SOME POWERS AND THEIR APPLICATIONS

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Abstract. In this paper we will discuss some powers of Baica & Cârdu Paratrigonometric Functions introduced for the first time by the authors in some previous papers when they developed some new Non-classical trigonometries. Also, we will emphasize their applications in Physics and Technology.

1. Introduction

In [2] the authors showed that the basic functions "Paratrigonometric sinus of α", denoted $spr_k\alpha$, has various values and consequently various graphical forms depending of the "order" value k. In Figure 1 is represented the function $spr_k\alpha$ for various k values such as: $k = 0.25$; $k = 0.5$; $k = 1.0$; $k = 2.0$; $k = 4.0$ and $k = \infty$.

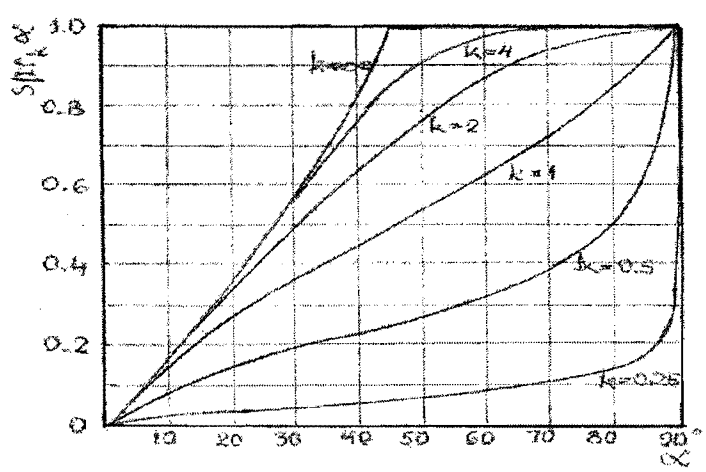

Fig.1: The paratrigonometric function $spr_k\alpha$ for 6 values of the "order" k

We recall that the fundamental relation in the Paratrigonometry is the following:

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$$(spr_k\,\alpha)^k + (cpr_k\,\alpha)^k = 1 \qquad (1.1)$$

Thus when $k = 2$, the relation (1.1) becomes the very well known relation of the Classical Trigonometry (CT) as such:

$$(\sin\,\alpha)^2 + (\cos\,\alpha)^2 = 1 \qquad (1.2)$$

The relations (1.1) and (1.2) indicate the fact that CT represents a particular case of the Paratrigonometry (PRT), when $k = 2$.

The "key" connection between PRT and CT is the following relation:

$$tpr_k\,\alpha = tg\,\alpha \qquad (1.3)$$

This relation is valid for any value of k.

From the relations (1.1) and (1.3) we can obtain the following relation for $spr_k\alpha$ which can be calculated from the function $tg\alpha$, very well known from the CT:

$$spr_k\,\alpha = tg\,\alpha\,/\,[1+(\mathrm{tg}\,\alpha)^k]^{1/k} \qquad (1.4)$$

Evidently that all the other paratrigonometric functions (cosine paratrigonometric, etc) can be calculated using the relations (1.1) and (1.4).

Returning to the Figure 1, we can see that in the spaces between the traced curves we can insert an infinite number of curves developed between the points with the coordinates (0^0; 0) and (90^0; 1.0), corresponding to the other values of k in the domain
$0 \leq k \leq \infty$.

All these curves "fill in" the space limited by Oα axis, the vertical line $\alpha = 90^0$ and the curve which represent the function $spr_\infty\alpha$. They all look as a "bunch of fibers" which start from a single point and terminate in the other, but each fiber has the form dictated by the corresponding $spr_k\alpha$ function respectively. We mention that the segment Oα, between the lines $\alpha = 0^0$ and $\alpha = 90^0$, together with the segment on the vertical line $\alpha = 90^0$, between the limits $spr_k\alpha = 0$ and $spr_k\alpha = 1.0$, represents $spr_k\alpha$ for $k = 0$, as we have shown in [1].

In order to represent some periodic functions with a sinusoidal like form, which we meet very often in Physics and Technology respectively, we consider that there is a need for other "traces" of the above mentioned "fibers". For this purpose we can apply to the function $spr_k\alpha$ raised to some powers, thing that raise very much the area of the paratrigonometric modeling of some periodic functions with sinusoidal forms. If we denote by p the power at which is raised the paratrigonometric function $spr_k\,\alpha$ we can write $(spr_k\,\alpha)^p$.

In Figure 2, as an example, we traced the curves which represent the function $(spr_k\,\alpha)^2$ and in Figure 3 we traced the curves which represent the function $(spr_k\,\alpha)^{0.5}$ for the same values of order k, as in Figure 1.

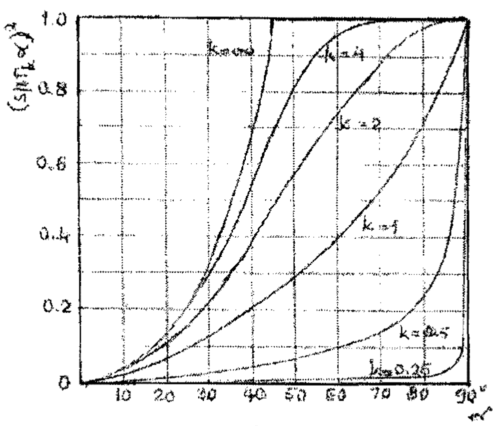

Fig.2

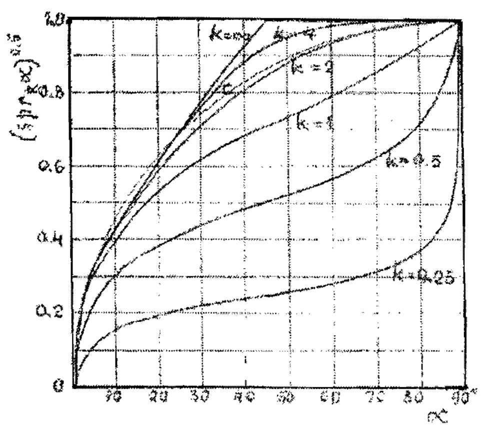

Fig.3

Fig. 2: The function $(spr_k\alpha)^2$ for the k values according the Figure 1.
Fig. 3: The function $(spr_k\alpha)^{0.5}$ for the k values according the Figure 1

Regarding the Figures 1, 2 and 3 we have to mention that by variation of k in the entire interval of possible values, such as $0 \leq k \leq \infty$ coupled with the variation of p in the entire interval of possible values, such as $0 \leq p \leq \infty$, we can cover the entire surface of the first trigonometric quadrant in the domain $0 \leq \alpha \leq 90^0$ and $0 \leq spr_k\alpha \leq 1.0$. We must stress the fact that any similarity of a sinusoidal periodic function $F(\alpha)$ with the function $(spr_k\alpha)^p$ when k and p have constant values, can be accepted when $F(\alpha)$ exactly coincides with $(spr_k\alpha)^p$ definite by the values of k and p respectively. In Physics and Technology we can meet with many such cases when they coincide, but in many situations $F(\alpha)$ can not be mathematically modeled to be equivalent with $(spr_k\alpha)^p$, where k and p have constant values. In these situations we can refuge to a mathematical modeling when k and p are variables. This modeling can't be done arbitrarily, but conform to some mathematical relations very well defined. About the power p, in the paper we consider both situations, when the power p is constant and other examples when p is variable as a function of α.

2. The case when the paratrigonometric function is raised to some constant powers.

We would like to mention that here, in order to simplify our analysis, in the Figures 1, 2 and 3, we used as a unit measure for the angle α, the old degree which is the case when the trigonometric circle has 360^0. Also, we would like to underline another important fact is that the representation scale for 90^0 (corresponding to the first trigonometric quadrant) on the abscissa axis was in this way chosen that this value would be equal with the dimension 1.0. The same dimensions would have to be equal to the maximum values of the functions $spr_k\alpha$ and $(spr_k\,\alpha)^p$ represented on the ordinate axis.

the algebraic point of view, by squaring the difference in the square parenthesis of these relations (2.2) and (2.3), they will become valid for the values of the angle α in this entire domain. For $180^0 \leq \alpha \leq 360^0$, y of relation (2.1) has negative values. Since the angle α in this interval has negative values will imply that *sinα* will be negative also and therefore to rise it to the power $p = 0.445$ is impossible.

In order to solve this problem we will write the relation (2.1) under the form:

$$y = \pm|sin\ \alpha|^{0.445} \qquad (2.4)$$

where

$|sin\ \alpha|$ is the absolute value of this function.

By the rules of the CT the sign +(plus) is used when the angle α is situated in the trigonometric quadrants I and II and the sign –(minus) is used when the angle α is situated in the trigonometric quadrants III and IV. We have to deal in the same way with the relations (2.3) and (2.4), but in this case for the values of α in the interval $180^0 \leq \alpha \leq 360^0$, based on the very well known relations of the CT, where we replace angle α by the angle $\beta = \alpha - 180^0$. Thus we have:

- for $0^0 \leq \alpha \leq 180^0$

$$y = \{1-[1-(\alpha/90)]^2\}^{1/2} \qquad (2.5)$$

$$sin\ \alpha = \{1-[1-(\alpha/90]^2\}^{1.1} \qquad (2.6)$$

- for $180^0 \leq \alpha \leq 360^0$

$$y = -\{1-[1-(\beta/90)]^2\}^{1/2} \qquad (2.7)$$

$$sin\alpha = -\{1-[1-(\beta/90])^2\}^{1.1} \qquad (2.8)$$

where $\beta = \alpha - 180^0$.

For $\alpha > 360^0$ the function y in the relation (2.4) will have a periodic variation of the function *sinα*, as we very well know from the CT. The graphical representation of the function y for the three semi periods (540^0 respectively 3π rad.) is given in Figure 4.

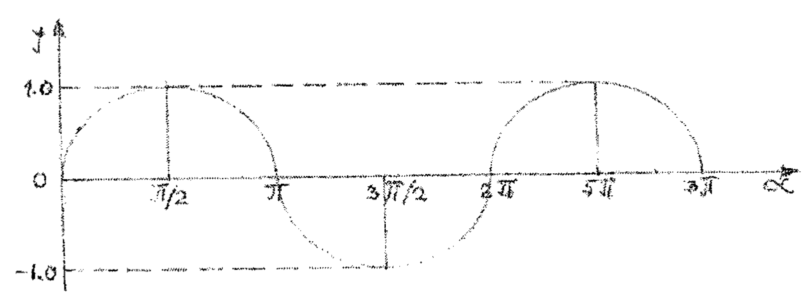

Fig. 4: The function y according the relations (2.5) and (2.7), for 3 semi periods (3π)

3. The case of a Paratrigonometric function rose to a variable power

It is known that the periodic functions of the "sinusoidal" type are amply used in Electro-technology. In very many cases these functions cannot be assimilated by sinusoids and having different forms they are represented with some mathematical expressions sometimes more complex and with a graphical representation which very often results from experiments.

As an example related with a modeling possibility of such graphical representation by a paratrigonometric function raised to a variable power we will choose the case of a periodic function, which appears in the Technology of the Electrical Machines. Thus, we refer to the curve of the current *i* in the rotor of an electrical machine when the case where the magnetic field between the teeth limiting the slots is considered – [3], page 102, Fig.76 a – We choose from this figure the case $U = 1.18U_n$. We considered that the maximal value for *i* is equal with the unit and we traced it in coordinates $[\alpha; (spr_k\alpha)^p]$ for $0 \leq \alpha \leq 90^0$ (Figure 5, curve "i").

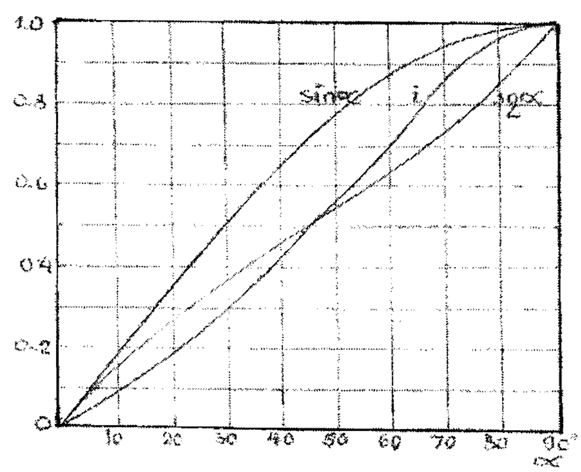

Fig. 5: Variation of *i* (defined as in the Chapter 3) and the paratrigonometric approximations of *i*

If we also trace in this figure the curve which represent the paratrigonometric function $spr_k\alpha$ of order $k = 1$, thus we see that $spr_1\alpha$ has a trace very close with the curve which represents the current *i* on one and the other of its sides. We recall that the order
$k = 1$ is characteristic to the Quadratic Trigonometry (QT) [2] and thus, we can replace
$spr_1\alpha$ by $sq\alpha$. The analyses of these two curves traces make us to deduce that the portion from the curve *i* function corresponding to some values of the angle α in the domain

$0^0 \leq \alpha \leq 47^0$ can be expressed by the function $sq\alpha$ raised to a supra unitary power ($p > 1$).

Contrary, the portion from the curve i corresponding to some values of the angle α in the domain $47^0 \leq \alpha \leq 90^0$ can be expressed by the function $sq\alpha$ raised to a under unitary power ($p < 1$). In this case we can write:

$$i = (sq\ \alpha)^p \qquad (3.1)$$

where p is, in this case, a variable value.

In order to determine the variation law of p as a function of α, for more of its values, we give p some diverse supra unitary values (for $\alpha < 47^0$) and under unitary values (for $\alpha > 47^0$) respectively, until the relation (3.1) is satisfied. This fact does not represent a problem when the computer is used. In the case of our example we performed these calculations for ten values of the angle α in the domain $0^0 \leq \alpha \leq 90^0$. We mention that these ten values for the angle α were chosen in the interval $5^0 \leq \alpha \leq 85^0$ since for $\alpha = 0$ we have $sq\alpha = 0$ and thus $i = 0$, and for $\alpha = 90^0$ we have $sq\alpha = 1$ and thus $i = 1$. The values of p obtained in this way are marked in a graph of $p = p(\alpha)$ in Figure 6 (with the reference to the ordinate axis on the left side).

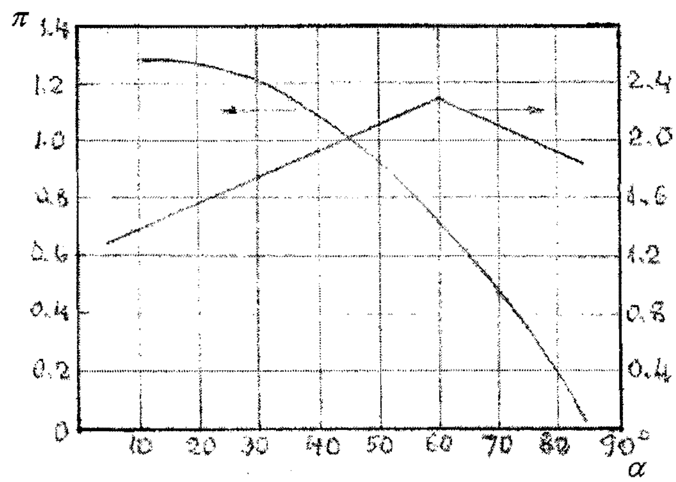

Fig. 6: Graphical representation of the function p according to (3.2) and of the function v according to (3.4)

Unifying the corresponding points we obtain the curve p which can be assimilated with a degree of precision sufficiently high with a parabola of second degree with the vertex at the point of coordinates ($\alpha = 10^0$; $p = 1.3$) and the ramifications developed towards the decreasing values of p. Having in view that this curve passes also through the point of the coordinates ($\alpha = 47^0$; $p = 1.0$), as we had shown above, we determined the definition elements of the parabola, arriving to the following relation for it:

$$p = 1.3 - 2.245.10^{-4}(\alpha - 10^0) \qquad (3.2)$$

Consequently it results that the variation curve of i as a function of α is mathematically modeled by the function $(sq\alpha)^p$ - see relation (3.1) - where p in its turn is a function of the angle α given by the relation (3.2). This mathematical model represents the variation curve of i conform [3] to a sufficient high precision, maximum error being of 5.6%.

Evidently that for $\alpha > 90^0$ the variation of i follows the trigonometric rules, starting from what we have shown above in regard with the corresponding function represented in the first trigonometric quadrant I.

The curve i can be modeled even if we admit that the function $(spr_2\alpha)^v$ which performs the modeling has v in function of the angle α, also. Since $spr_2\alpha = sin\alpha$, we have:

$$i = (\sin \alpha)^v \qquad (3.3)$$

Proceeding in the same way as above, we established values for v corresponding to a sufficiency large number of the values for the angle α. Uniting the points of coordinates (α;v) we get with a sufficient precision these two concurrent lines in the point
($\alpha = 60^0$; $v = 2.3$), represented in Figure 6 (referred to the ordinate axis on the right side).
These lines are represented by the equation:

$$v = 2.3 - 0.018|60^0 - \alpha| \qquad (3.4)$$

Calculating the values of the function i using the relations (3.3) and (3.4) we obtain the deviations of its values from the graph we intend to modeling of maximum 4.8%.

Using the examples discussed in this chapter we can see that there exists multiple possibilities to mathematically model some functions defined graphically by applying the paratrigonometric functions raised to some powers, generic written by $(spr_k\alpha)^p$, where k can have values in a very wide range and the power p in its turn can have diverse mathematical expressions under the form of some functions $p(\alpha)$.

4. Conclusions

The following important conclusions summarize what we have discussed in these above-mentioned chapters:

4.1. The paratrigonometric functions, especially the function $spr_k\alpha$, have multiple capabilities to model functions graphically defined which are frequently met in Physics and Technology, respectively. This thing is due the fact that the "order" k of the paratrigonometric function can have infinitely many values. These capabilities of the paratrigonometric function $spr_k\alpha$ can be more multiplied if the corresponding function is raised to a power p. In this case we have the function $(spr_k\alpha)^p$.

4.2. Many graphically defined functions and some algebraically defined functions can be modeled by paratrigonometric functions raised to a constant power. Such a case is represented in the modeling of the circle with a high precision.

4.3. A large possibility to model some functions graphically defined which are frequently met in Physics and in Technology is to use the paratrigonometric functions raised to some powers which themselves are defined as functions of the angle α. In Chapter 3 we discussed the case of a function graphically.

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