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REGARDING THE GENERAL CHARACTER OF THE POLYGONAL TRIGONOMETRY

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Abstract. In this paper we will demonstrate the general character of the polygonal trigonometry (PT) whose fundamental elements are valid for both the case of quadratic trigonometry (QT) and the case of classical trigonometry (CT).

QT is the particular case of the PT situated at the lower limit of the number of sides n of trigonometric polygon ($n = 4$), and CT is the particular case of PT situated at the upper limit of the value n, thus $n = \infty$.

Keywords and phrases: Quadratic trigonometry (QT), Polygonal trigonometry (PT), Classical trigonometry (CT).

1. Introduction

In their paper [2] the authors presented the basics of the polygonal trigonometry (PT) together with the fundamental relations of PT starting from the foundations of the quadratic trigonometry (QT) elaborated by professor V. Alaci of the "Politehnica" University of Timişoara, Romania, in the 1930's.

Distinct from the classical trigonometry (CT) which is based on the trigonometric circle with center at O and radius one, QT is developed on a trigonometric square inscribed in a circle with $r = 1$, having its corner at the angles, 0 (zero), $\frac{\Pi}{2}$, Π, $\frac{3\Pi}{2}$, 2Π expressed in radians (respectively 0°, 90°, 180°, 270° and 360° expressed in degrees) of the trigonometric circle.

Similarly, the fundamental relations of PT presented in [2], were established based on a regular trigonometric polygon inscribed in a circle with $r = 1$. In order to maintain the symmetry conditions this trigonometric polygon must have a number of sides equal to a multiple of four, thus

$$n = 4 \cdot m \tag{1.1}$$

where n is the number of sides of the trigonometric polygon, and m is a positive integer.

Just as for trigonometric square, the corners of the trigonometric polygon after $n/4$, $n/2$ and $3\,n/4$ sides are situated at the angles 0, $\frac{\Pi}{2}$, Π, $\frac{3\Pi}{2}$, of the circle with $r = 1$ circumscribed about the trigonometric polygon.

We make the observation that the trigonometric square is a trigonometric polygon with the minimum possible number of sides ($n = 4$ implies $m = 1$).

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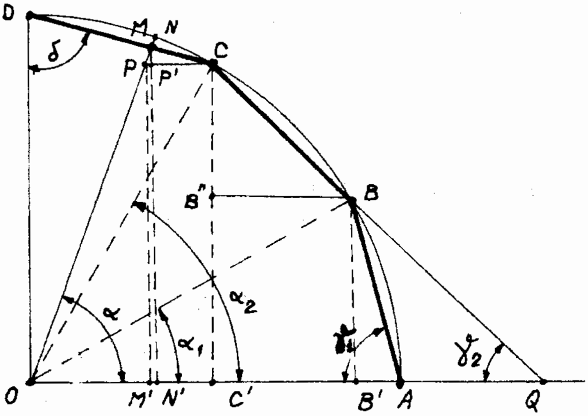

Fig. 1

Refering to Figure 1 where we represent the first quadrant of a trigonometric polygon with 12 sides, $n = 12$ and thus $m = 3$, in the paper [2] we established the following fundamental relation in PT, valid also for a trigonometric polygon with any number of sides n (respecting $n = 4 \cdot m$):

$$cp(n)\alpha + \frac{sp(n)\alpha - \sin(i-1)\frac{2\Pi}{n}}{\mathrm{tg}\,\gamma_i} + \sum_{j=1}^{j=i-1} \frac{\sin\left[(i-j)\frac{2\Pi}{n} - \sin(i-j-1)\frac{2\Pi}{n}\right]}{\mathrm{tg}\,\gamma_{i-j}} = 1 \qquad (1.2)$$

where $cp(n)\alpha$ is the polygonal (p) cosines of α function for the trigonometric polygon with n sides, $sp(n)\alpha$ is the polygonal (p) sines of a function respectively, i is the current number of the circular sector where angle α is situated, counted in the trigonometric direction starting from $\alpha = 0$, and

$$\gamma_i = (n - 4\,i + 2)\frac{\Pi}{2n} \qquad (1.3)$$

$$\gamma_{i-j} = [n + 4(i-j) + 2] \qquad (1.4)$$

Another important relation valid in all CT, QT and PT is the one which defines the tangent of α function:

$$\mathrm{tg}\,\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{sq\alpha}{cq\alpha} = \frac{sp(n)\alpha}{cp(n)\alpha} \qquad (1.5)$$

where sq(α) and cq(α) are quadratic sines and quadratic cosines of a functions respectively. Thus it follows:

$$tp(n)\alpha = tq(\alpha) = \mathrm{tg}\,\alpha \qquad (1.6)$$

In the paper [2] we showed that applying formula (1.2) in the trigonometric square (n = 4) case we obtain the fundamental relation of QT, namely

$$sq\,\alpha + cq\,\alpha = 1 \qquad (1.7)$$

In what follows, this paper will analyze the manner in which the basic elements of PT are applied in the CT case, which in fact represents a particular case of PT (when $n = \infty$), as well as in the QT case which represents the other extremes value of n ($n = 4$).

Evidently, $n_{\min} = 4$ and $n_{\max} = \infty$.

2. Classical trigonometry (CT) a particular limiting case of the polygonal trigonometry (PT)

For the above mentioned analysis we start using Figure 1 and the geometric elements from this figure, with the help of which we obtain the fundamental relation (1.2) applied in the PT. Thus, in the Figure 1 we see that for the triangles OB'B and OC'C having the sharp corners B and C of the trigonometric polygon situated on the trigonometric circle, the Pythagorean theorem gives the relations:

$$\left(\overline{OB'}\right)^2 + \left(\overline{BB'}\right)^2 = 1$$

$$\left(\overline{OC'}\right)^2 + \left(\overline{C'C}\right)^2 = 15$$

In the CT (with reference to the trigonometric circle) we have $\overline{OB'} = \cos \alpha_1$ and $\overline{B'B} = \sin \alpha_1$ and respectively, $\overline{OC'} = \cos \alpha_2$ and $\overline{C'C} = \sin \alpha_2$ and thus:

$$\cos^2 \alpha_1 + \sin^2 \alpha_1 = 1 \qquad (2.3)$$

$$\cos^2 \alpha_2 + \sin^2 \alpha = 1 \qquad (2.4)$$

But the trigonometric circle is a trigonometric polygon with an infinite number of sides.

Thus all the points which form this circle could be considered as sharp corners (as well as B and C) of the trigonometric polygon with $n = \infty$. In other words, in the trigonometric circle case, applied to CT, the relations (2.1) and (2.2) and respectively (2.3) and (2.4) are valid for any point on the circle. Therefore, regarding the current angle α, we have:

$$\cos^2 \alpha + \sin^2 \alpha = 1 \qquad (2.5)$$

This (2.5) is a fundamental relations of CT.

It follows then that the trigonometric circle represents the upper limit (for $n = \alpha$) of the trigonometric polygon. Consequently, CT represents a particular case (for $n = \infty$) of the PT.

3. The general character of the polygonal trigonometry (PT)

From the previous section of this paper and from paper [2] it follows that PT is generally applicable; the mathematical elements which guide us to its fundamental relation make it valid for both QT and CT.

QT is situated at the lower (inferior) limit of the number of sides of the trigonometric polygon ($n = 4$), and CT represents the upper (superior) limit of the PT from this point of view ($n = \infty$).

The fundamental relations of PT have a general character and those of QT and CT represent particular cases of the PT, and they are as follows:

- QT .... relation (1.7)
- PT .... relation (1.2)
- CT .... relation (2.5)

The relations (1.7) and (2.5) can also be written as follows:

$$\cos^k \alpha + \sin^k \alpha = 1 \qquad (3.1)$$

where k = 1 for QT and respectively k = 2 for CT.

Considering the above mentioned facts, logically it appears that the formula (3.1) could also be valid in the PT, the exponent valve k varying in the closed interval ($1 \leq k \leq 2$) and depending on the value of n which characterizes the corresponding trigonometric polygon.

To investigate the validity of such a hypothesis we proceeded to calculated the values of k as a function of the angle α for three distinct values of n from the range $4 < n < \infty$, these values being $n = 8$, $n = 16$, and $n = 24$.

For this reason we use formulas (1.6) and (3.1) and we have:

$$[cp(n)\alpha]^k = \frac{1}{1+(tg\,\alpha)^k} \tag{3.2}$$

k is part of the formula (3.2) and as such in order to determinate its values we give k different values in the domain $1 \le k \le 2$. From equation (3.2) using logarithms we obtain the following relation for cp (α):

$$cp(n)\alpha = e^z \tag{3.3}$$

where

$$z = \frac{\ln \mathbf{R}}{\mathbf{k}} \tag{3.4}$$

$$\mathbf{R} = \frac{1}{1+(tg\,\alpha)^k} \tag{3.5}$$

Value of cp(n)α resulting from formula (3.3) is then compared with the one calculated with the exact relation for this function resulting from formula (1.2) where sp(n)α is replaced with the right side term of the equality

$$sp(n)\alpha = [cp\ (n)\ \alpha] \times tg\ \alpha \tag{3.6}$$

With successive trials we obtain an exact value for k. In this way we find that k depends not only on n but also on the value of the angle α.

The variation of k as a function of α, for these three values of n, namely $n = 8$, $n = 16$ and $n = 24$, is graphically represented in Figure 2.

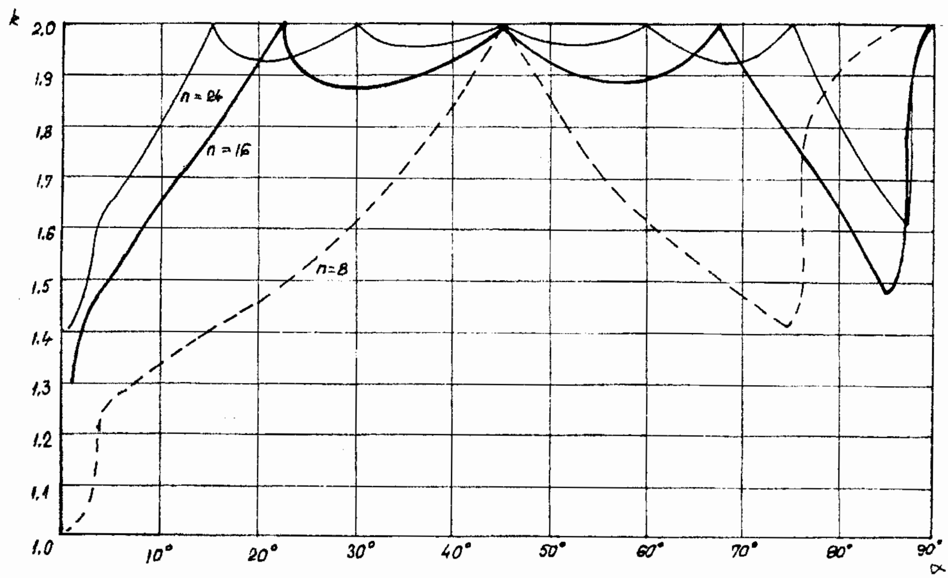

Fig.2

We have calculated the average values of k for all values of the angle a in the first quadrant of the trigonometric circle. The values thus obtained for k_a in these three analyzed cases

(n = 8, n = 16 and n = 24) are contained in Table 1, where we give synthetically the general character of the PT.

Tab. 1

n	4	8	16	24	...	∞
The Trigonometry	QT		PT			CT
Fundamental relations	(1.7)		(1.2)			(2.5)
	(1.5)		(1.5)			(1.5)
k	4	k_a = (1.673)	k_a = (1.822)	k_a = (1.901)	...	2

In Table 1, the values of k_a were given in parentheses since they can not be practically used in formula (3.1) which for $4 < n < \infty$ is proved to be only hypothetical.

The value of k is constant for any value of α, only for n = 4 (k = 1) and $n = \infty$ (k = 2).

In Figure 3 we represent the values of k (for n = 4) - point A - and k_a for n = 8, n = 16 and n = 24 (points B, C, D), and the curve which unites these points.

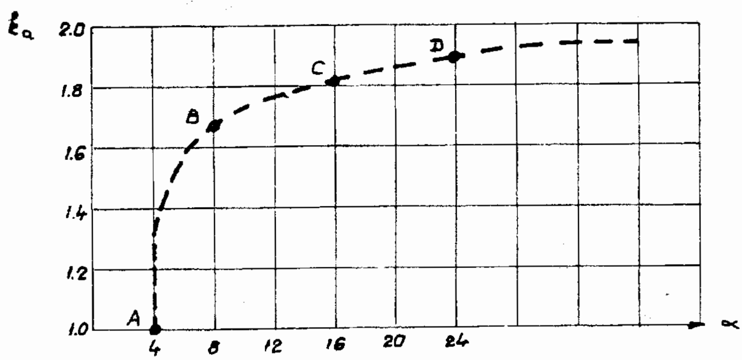

Fig.3

This curve was traced with an interrupted line, since it is not continuous.

It only units some single points (A, B, C, D, ...) characterized by abscissa values $n = 4 \cdot m$, where n is a positive integer, as we said at the beginning.

The curve ABCD . . . of Figure 3 is interesting because it reveals a sharp increase of k_a from n = 4 to n = 8, after which k_a increases slowly with the increase of n, tending asymptotically toward k = 2 for $n=\infty$.

4. Conclusions

The polygonal trigonometry (PT) whose basic elements were presented in paper [2] has a general character such that the quadratic trigonometry (QT) [1] and the classical trigonometry (CT) represent the limiting cases for PT.

QT is derived from the PT, when the number of sides n of the trigonometric polygon have a minimum value n = 4, and CP is derived from the PT when n has the maximum value $n = \infty$.

The fundamental relation of the QT is (1.7) and in the CT the fundamental relation is (2.5) both of these relations are of the form of relation (3.1) such as

$$\cos^k \alpha + \sin^k \alpha = 1$$

In the extreme cases of the value of n, the k exponent has constant values as a function of the angle α. Thus, for $n = 4$ (QT), k - 1, and for $n = \infty$ (CT), k = 2.

For all the other cases ($4 < n < 8$), the values of k are functions of both n and α. The k exponent varies in these cases, as a function of n and α in the interval $1 < k < 2$. For each value of n in the interval $4 < n < \infty$, the k exponent attains the value k = 2 at the sharp corners of the trigonometric polygon (see Figure 1) that is, for values of the angle α (in radians) which equal $\left(\frac{2\Pi}{n}\right) i$, where i is the current number of circle sectors determined by the trigonometric polygon.

The average values of k, denoted by k_a, which do not have a practical application, show its tendency to increase with the increase of n, from k = 1 (for $n = 4$) to k = 2 (for $n = \infty$).

The list of figures and tables

References

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