# BAICA'S CONTRIBUTION TO THE SOLUTION OF THE BINARY GOLDBACH PROBLEM

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**Abstract**

In this paper, some commentaries are given regarding Baica's previous papers [1, 2] on the proof of Goldbach's Conjecture. Therefore our target is now to give a brief account or survey of [1, 2].

## 1 Introduction

On June $17^{th}$, 1742, in a letter to Euler, Goldbach stated: "that every even integer is the sum of two integers $p$ and $q$ where each of $p$ and $q$ are either one or odd primes." In other words, that every integer $n$ greater than five is the sum of three primes. Euler replied to Goldbach: "There is little doubt that this result is true, but that every even number is a sum of two primes, I consider an entirely certain theorem in spite of that I am not able to demonstrate it." Therefore we have the Goldbach's Conjecture: Every even integer $n$ greater than two is the sum of two primes. Descartes was the first who knew the two prime version of this conjecture before Goldbach did. Erdös said that conjecture is misnamed and it "is better that the conjecture be named after Goldbach because mathematically speaking, Descartes was infinitely rich and Goldbach was poor."

We know that Goldbach's problem admits two variants:

a) Every odd number greater than five is the sum of three odd primes.

b) Every even number greater than two is the sum of two odd primes.

Variant a) is called the “Ternary” Goldbach problem, and recently, Jean-Marc Deshouillers, Yannick Saouter and Herman Teriele have verified this up to $10^{14}$ with the help of a Cray C90 and various workstations (Ref. [3]). It is important to say that under extended Riemann Hypothesis (ERH) assumption this is enough to prove the odd Goldbach conjecture.

Variant b) is called the “Binary” Goldbach problem and nothing really important was proved about it until 2001 and 2002, when Ref. [1, 2] were published. Eighty years of experience has shown that b) is much more difficult to prove than a). In 1923, Hardy and Littlewood [4] took the first major step toward the proof of the Goldbach's conjectures using their circle method. In 2001, the author [1], in order to determine the error of the Hardy-Littlewood formula which can not be done by the circle method, used for the first time the Laplace transform as a principal and crucial tool.

# 2 Survey of Baica's Proof of Goldbachs Conjecture (Ref. [2])

Note: in what follows, we assign to each formula the same numbering systm that was used in Ref. [2].

In their fundamental paper on the problem (Ref. [4]), Hardy and Littlewood used the function:

$$v(t) = \sum_{p_1+p_2=t} \log p_1 \cdot \log p_2$$

where the $p_i$ $(i = 1, 2)$ stand for the prime numbers.

This function is connected with the Chebyshev function $\theta(u)$

$$\theta(u) = \sum_{p \leq u} \log p$$

through the relation:

$$v(t) = \sum_{u<t} \{\theta(u) - \theta(u-1)\} \{\theta(t-u) - \theta(t-u-1)\}$$

The right hand side can be transformed to an integral, and thus we get that:

**(3.4)**
$$v(t) = \int_0^t \Delta\theta(u+1)\Delta\theta(t-u)du$$

where

$$\Delta\theta(u) = \theta(u) - \theta(u-1)$$

(3.4) is an integral of convolutory type, and as known, it can be dealt suitably with the Laplace transform $L$.

Thus we obtain:

$$L\{v(t)\} = e^s L^2 \{\Delta\theta(u)\} \tag{3.6}$$

For $L\{\Delta\theta(u)\}$ we have the formula

$$L\{\Delta\theta(u)\} = \frac{1-e^{-s}}{s} \sum_p \log p \cdot e^{-ps} \tag{4.2}$$

(This saves a misprint in Ref. [2]).

Now we need to know the right hand function regarding the infinitude of singularities that it has on the line $\sigma = 0$.

This question was solved by Hardy and Littlewood in Ref. [4], where they found that:

$$\sum \log p \cdot e^{-ps} = \sum_{q=1}^{[\sqrt{n}]} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} \frac{\mu(q)}{\varphi(q)(s + 2\pi i h/q)} + An^{\theta+1/4+\varepsilon} \tag{4.4}$$

Replacement of this in (4.2) leads to the formula

$$L\{v(t)\} = e^s \left(\frac{1-e^{-s}}{s}\right)^2 \left\{ \sum_{q=1}^{[\sqrt{n}]} \sum_{h=1}^{q} \frac{\mu(q)}{\varphi(q)(s + 2\pi i h/q)} + An^{\theta+1/4+\varepsilon} \right\}^2 \tag{5.1}$$

Formula (4.4) was proved by Hardy-Littlewood assuming the ERH (Extended Riemann Hypothesis). A short time after [2] was published, Peretti succeeded in proving it and the proof will be soon published in internet, as a straight forward generalization of the several proofs he has developed for the Riemann Hypothesis which also will be published in the internet. See, for instance, Ref. [5]. Hence, formula (5.1) is valid with $\theta = \frac{1}{2}$.

# 3 An Immediate Consequence of (5.1)

$$\frac{v(t-1+0) + v(t-1-0)}{2} = \tag{5.2}$$

$$L^{-1}\left\{\left(\frac{1-e^{-s}}{s}\right)^2 \sum_{q=1}^{[\sqrt{n}]}\sum_{h=1}^{q}\frac{\mu^2(q)}{\varphi^2(q)(s+2\pi ih/q)^2}+\left(\frac{1-e^{-s}}{s}\right)^2\cdot\right.$$

$$\sum_{q_1}^{[\sqrt{n}]}\sum_{h_1}^{q_1}\sum_{q_2}^{[\sqrt{n}]}\sum_{h_2}^{q_2}\frac{\mu(q_1)\mu(q_2)2An^{\frac{3}{4}+\varepsilon}}{\varphi(q_1)\varphi(q_2)(s+2\pi ih_1/q_1)(s+2\pi ih_2/q_2)}+$$

$$\left.\left(\frac{1-e^{-s}}{s}\right)^2 An^{\frac{3}{2}+2\varepsilon}\right\}$$

From this, after a lengthy but straight-forward calculation, one arrives at the following result:

**(12.2)**
$$\begin{aligned} v(t) = {} & P(t)(t+1)+ \\ & 2\delta_3\delta_4 e^{3\gamma} d(t)\frac{(\log\log N)^2}{N}\log\log(t+1-\delta_5)- \\ & \delta_2 N^2(N+1)^2 AN^{\frac{3}{2}+\varepsilon}+2A_1N^{3+\varepsilon} \end{aligned}$$

Here $P(t)$ is the infinite product

$$P(t)=2\prod_{p=3}^{\infty}\left(1-\frac{1}{(p-1)^2}\prod_{p/t}\frac{p-1}{p-2}\right)$$

The $\delta_i$ are such that $|\delta_i|<1$; $d(t)$ = number of divisors of $t$; $\gamma$ = Euler's constant and $t$ must be assumed to be an even number. $A$ and $A_1 < 80$.

# 4 Judicious choice of $N$ as $N = t^{\frac{1}{6.5}}$ in (12.2) leads to the conclusion that:

**(13.3)**
$$v(t)=P(t)t-2e^{3\gamma}d(t)(\log\log t)^3 t^{1-\frac{1}{7.5}}-\frac{80}{2\pi}(t+1)^{\frac{4}{5}}-60t^{\frac{5+2\varepsilon}{7.5}}$$

But if there is only one solution in the decomposition $t = p_1 + p_2$ we have that

**(14.3)**
$$v(t)=\frac{1}{2}\log^2\frac{t}{2}$$

From (13.3) and (14.3) it can be deduced that the Goldbach's Conjecture is valid for even $t > 10^B$ with $180 < B < 190$. This is the better (and only) explicit general result obtained up to the present in the binary Goldbach problem. The original Hardy-Littlewood circle method does not allow to obtain such a bound.

## References

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