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SOME NEW COMBINATORIAL IDENTITIES DERIVED FROM UNITS IN ALGEBRAIC NUMBER FIELDS

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Dedicated to the memory of Jürgen Schmidt*

Two new combinatorial identities are derived from explicitly stated units in algebraic number fields of degree $n=3$. Let $e=w-D$ be a unit in a cubic field where $w^3=D^3+1$, $D\in\mathbb{N}$ and $n=0,1,\ldots$. Then $z_n=t_{n+1}^2-t_nt_{n+2}$ and $t_n=z_{n-1}^2-z_{n-2}z_n$ are two combinatorial identities where t_{n+2} and z_{n+1} are obtained from some recursion formulas. Both have the same structure.

Introduction

One could be surprised why such a mathematical field as combinatorics should be coupled with the units in algebraic number fields. The connection between these two, seemingly entirely different domains of mathematics, rests upon the fact that in two papers [2, 3] Bernstein invented a method from which combinatorial identities can be derived from explicitly stated units in algebraic number fields of any degree $n\geq 2$. It is not really necessary to make use of the theory of units to produce such results since Carlitz [5, 6, 7], who is the master of these results, obtained Bernstein's combinatorial identities and many other more difficult ones using the classical method. This method is interesting since, sometimes Bernstein's method is capable of producing some results with simpler computations. The difference between the combinatorial identities established by Bernstein [4] and those generated in this paper rests, of course, with the choice of units.

We choose our units from the set of units given by

$$e=\frac{(w-D)^s}{w^s-D^s},\quad 1<s\leq n,\ w^n=D^n+d,\ d\mid D,\ D\in\mathbb{N},\ d\in\mathbb{Z},\ s\mid n.$$

The author derived these units from the periodicity of ACF developed in [1].

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1. Statement of the problem

Theorem 1. *Let $e = w - D$ be a unit in a cubic field where $w^3 = D^3 + 1$, $D \in \mathbb{N}$, and $n = 0, 1, \ldots$. Then*

$$z_n = t_{n+1}^2 - t_n t_{n+2} \quad \text{and} \quad t_n = z_{n-1}^2 - z_{n-2} z_n$$

are two combinatorial identities where t_{n+2} and z_{n+1} are as follows:

$$t_{n+2} = \sum_{\substack{x_1+x_2+x_3=i \\ 3x_1+2x_2+x_3=n}} (-1)^{x_2+x_3} \binom{i}{x_1, x_2, x_3} 3^{x_2+x_3} D^{2x_2+x_3},$$

and

$$z_{n+1} = \sum_{\substack{y_1+y_2+y_3=i \\ 3y_1+2y_2+y_3=n}} \binom{i}{y_1, y_2, y_3} 3^{y_2+y_3} D^{y_2+2y_3}.$$

2. Positive powers of e

In this section we shall calculate positive powers of $e = w - D$, taking in consideration that $w^3 = D^3 + 1$:

$$\begin{aligned}
e^0 &= 1 = 1 + 0 \cdot w + 0 \cdot w^2, \\
e &= w - D = -D + 1 \cdot w + 0 \cdot w^2, \\
e^2 &= (w-D)^2 = D^2 - 2Dw + 1 \cdot w^2, \\
e^3 &= (w-D)^3 = 1 + 3D^2 w - 3Dw^2, \\
e^4 &= (w-D)^4 = (-3D^4 - 4D) - (3D^3 - 1)w + 6D^2 w^2, \\
e^5 &= (w-D)^5 = 9D^5 + 10D^2 - 5Dw - (9D^3 - 1)w^2.
\end{aligned}$$

We now denote

$$e^n = (w-D)^n = r_n + s_n w + t_n w^2, \quad n = 0, 1, \ldots, r_n, s_n, t_n \in \mathbb{Z}. \tag{2.1}$$

From our previous calculations we have

$$\begin{array}{lll}
r_0 = 1, & s_0 = 0, & t_0 = 0, \\
r_1 = D, & s_1 = 1, & t_1 = 0, \\
r_2 = D^2, & s_2 = -2D, & t_2 = 1, \\
r_3 = 1, & s_3 = 3D^2, & t_3 = -3D, \\
r_4 = -3D^4 - 4D, & s_4 = -3D^3 + 1, & t_4 = 6D^2, \\
r_5 = 9D^5 + 10D^2, & s_5 = -5D, & t_5 = -9D^3 + 1.
\end{array} \tag{2.2}$$

We denote

$$w^3 = D^3 + 1 = m \tag{2.3}$$

and multiply both sides of (2.1) by $e = w - D$, we obtain

$$\begin{aligned} e^{n+1} &= r_n(w-D)+s_n(w^2-Dw)+t_n(w^3-Dw^2) \\ &= mt_n - Dr_n + (r_n - Ds_n)w + (s_n - Dt_n)w^2. \end{aligned} \tag{2.4}$$

From (2.1) and (2.4) we have

$$r_{n+1} = mt_n - Dr_n, \tag{2.5a}$$

$$s_{n+1} = r_n - Ds_n, \tag{2.5b}$$

$$t_{n+1} = s_n - Dt_n. \tag{2.5c}$$

From (2.5c) we obtain

$$s_n = t_{n+1} + Dt_n. \tag{2.6}$$

From (2.5b) and (2.6) we obtain

$$r_n = t_{n+2} + 2Dt_{n+1} + D^2 t_n. \tag{2.7}$$

From (2.5a) and (2.7) we obtain

$$\begin{aligned} t_{n+3}+2Dt_{n+2}+D^2t_{n+1} &= mt_n - D(t_{n+2}+2Dt_{n+1}+D^2t_n) \\ &= (D^3+1)t_n - Dt_{n+2} - 2D^2t_{n+1} - D^3t_n \end{aligned}$$

$$t_{n+3} = t_n - 3D^2 t_{n+1} - 3Dt_{n+2}. \tag{2.8}$$

Substituting the values of s_n, r_n from (2.6), (2.7) in (2.1)

$$\begin{aligned} &e^n = t_{n+2}+2Dt_{n+1}+D^2t_n+(t_{n+1}+Dt_n)w+t_nw^2, \\ &t_{n+3} = t_n - 3D^2t_{n+1} - 3Dt_{n+2}. \end{aligned} \tag{2.9}$$

(2.9) expresses e^n by one parameter t_n recursively given and w. For $t_0 = t_1$, $t_2 = 1$ as in (2.2) we calculate t_n by using Euler's generating function:

$$\begin{aligned} \sum_{n=0}^{\infty} t_n u^n &= t_0u_0 + t_1u + t_2u^2 + \sum_{n=3}^{\infty} t_n u^n \\ &= u^2 + \sum_{n=0}^{\infty} t_{n+3}u^{n+3} \\ &= u^2 + u^3 \sum_{n=0}^{\infty} (t_n - 3D^2t_{n+1} - 3Dt_{n+2})u^n \\ &= u^2 + u^3 \sum_{n=0}^{\infty} t_nu^n - 3D^2u^2 \sum_{n=0}^{\infty} t_{n+1}u^{n+1} - 3Du \sum_{n=0}^{\infty} t_{n+2}u^{n+2} \\ &= u^2 + u^3 \sum_{n=0}^{\infty} t_nu^n - 3D^2u^2\left[\left(\sum_{n=0}^{\infty} t_nu^n\right) - t_0u^0\right] \\ &\quad - 3Du\left[\left(\sum_{n=0}^{\infty} t_nu^n\right) - t_0u^0 - t_1u\right], \end{aligned}$$

and since $t_0 = t_1 = 0$,

$$\sum_{n=0}^{\infty} t_n u^n = u^2 + u^3 \sum_{n=0}^{\infty} t_n u^n - 3D^2u^2 \sum_{n=0}^{\infty} t_n u^n - 3Du \sum_{n=0}^{\infty} t_n u^n,$$

$$(1-u^3+3D^2u^2+3Du) \sum_{n=0}^{\infty} t_n u^n = u^2,$$

$$\sum_{n=0}^{\infty} t_n u^n = \frac{u^2}{1-(u^3-3D^2u^2-3Du)},$$

$$t_0u^0 + t_1 u + \sum_{n=2}^{\infty} t_n u^n = \frac{u^2}{1-(u^3-3D^2u^2-3Du)},$$

$$\sum_{n=0}^{\infty} t_{n+2} u^{n+2} = \frac{u^2}{1-(u^3-3D^2u^2-3Du)},$$

$$\sum_{n=0}^{\infty} t_{n+2} u^n = \frac{1}{1-(u^3-3D^2u^2-3Du)}. \tag{2.10}$$

We choose u sufficiently small such that

$$|u^3-3D^2u^2-3Du|<1. \tag{2.11}$$

It suffices to choose $|u|<1/9D$

$$|u^3-3D^2u^2-3Du| \leq |u^3| + 3D^2|u^2| + 3D|u|$$
$$< \frac{1}{729D^3} + \frac{3D^2}{81D^3} + \frac{3D}{9D} < \frac{1}{3D} + \frac{1}{3D} + \frac{1}{3} < 1.$$

Since $D>1$ we expressed the right-hand side of (2.10) as an infinite, absolutely convergent series and obtain:

$$\sum_{n=0}^{\infty} t_{n+2} u^n = \sum_{j=0}^{\infty} (u^3-3D^2u^2-3Du)^j. \tag{2.12}$$

We find t_{n+2} $(n=1, 2, \ldots)$ from (2.12) by comparison of coefficients. Now, we are looking for the coefficients of u^n in the expansion on the right-hand side. Here the exponent j varies from 0 to ∞. Since the polynomial which is being raised to the power j has the highest power u^3, the exponent j, in order to obtain u^n cannot be smaller than $[\frac{1}{3}n]$ and since this polynomial has the smallest power u, the exponent j cannot be larger than n. Thus in order to obtain all powers of u^n in the expansion of the right-hand side of (2.12) we have to investigate the sum

$$\sum_{i=[n/3]}^{n} (u^3-3D^2u^2-3Du)^i. \tag{2.13}$$

We expand the polynomial in (2.13) by the multinomial formula and obtain

$$(u^3-3D^2u^2-3Du)^i = \sum_{x_1+x_2+x_3=i} \binom{i}{x_1, x_2, x_3} (u^3)^{x_1}(-3D^2u^2)^{x_2}(-3Du)^{x_3} \tag{2.14}$$

or

$$(u^3-3D^2u^2-3Du)^i=\sum_{x_1+x_2+x_3=i}\binom{i}{x_1,\,x_2,\,x_3}(-1)^{x_2+x_3}3^{x_2+x_3}D^{2x_2+x_3}u^{3x_1+2x_2+x_3}. \tag{2.15}$$

Now we are looking for elements u^n and we have to set

$$3x_1+2x_2+x_3=n. \tag{2.16}$$

From the two conditions

$$x_1+x_2+x_3=i, \qquad 3x_1+2x_2+x_3=n, \tag{2.17}$$

and the bounds of i, we finally have

$$t_{n+2}=\sum_{i=[n/3]}^{n}\ \sum_{\substack{x_1+x_2+x_3=i\\ 3x_1+2x_2+x_3=n}}(-1)^{x_2+x_3}\binom{i}{x_1,\,x_2,\,x_3}3^{x_2+x_3}D^{2x_2+x_3}. \tag{2.18}$$

(2.18) is the formula which states t_n in an explicit form.

The bounds of i are already given by the two restrictions (2.17), so that formula (2.18) can also be written as

$$t_{n+2}=\sum_{\substack{x_1+x_2+x_3=i\\ 3x_1+2x_2+x_3=n}}(-1)^{x_2+x_3}\binom{i}{x_1,\,x_2,\,x_3}3^{x_2+x_3}D^{2x_2+x_3}. \tag{2.19}$$

3. Negative powers of e

This section is devoted to the calculation of the negative powers of e. Since this problem is essentially the same as the finding of the positive powers in the previous section, we shall carry out many operations without giving again the necessary explanations. We shall set out with calculating a few initial values of e^{-1}, with $e=w-D$, $w^3=D^3+1$, we have $w^3-D^3=1$, $(w-D)(w^2+Dw+D^2)=1$.

$$e^{-1}=(w-D)^{-1}=\frac{1}{w-D}=D^2+Dw+w^2, \tag{3.1}$$

$$e^{-2}=(D^2+Dw+w^2)^2=3D^4+2D+(3D^3+1)w+3D^2w^2.$$

Denoting

$$e^{-n}=x_n+y_nw+z_nw^2, \quad n=0,1,\ldots, x_n, y_n, z_n\in\mathbb{Z}, \text{ for } n>0, \tag{3.2}$$

we have for the initial values

$$\begin{aligned}
&x_0=1, && y_0=0, && z_0=0,\\
&x_1=D^2, && y_1=D, && z_1=1,\\
&x_2=3D^4+2D, && y_2=3D^3+1, && z_2=3D^2.
\end{aligned} \tag{3.3}$$

From (3.2) we obtain, multiplying both sides by $e^{-1}=D^2+Dw+w^2$,

$$e^{-(n+1)}=D^2x_n+(D^3+1)y_n+D(D^3+1)z_n$$
$$+[Dx_n+D^2y_n+(D^3+1)z_n]w+(x_n+Dy_n+D^2z_n)w^2.$$

Hence by definition

$$x_{n+1}=D^2x_n+(D^3+1)y_n+D(D^3+1)z_n, \tag{3.4a}$$

$$y_{n+1}=Dx_n+D^2y_n+(D^3+1)z_n, \tag{3.4b}$$

$$z_{n+1}=x_n+Dy_n+D^2z_n. \tag{3.4c}$$

From (3.4) we obtain, making the operations (3.4a)$-D$(3.4b), (3.4b)$-D$(3.4c),

$$x_{n+1}-Dy_{n+1}=y_n, \qquad y_{n+1}-Dz_{n+1}=z_n. \tag{3.5}$$

From (3.5)

$$x_{n+1}=y_n+Dy_{n+1}, \qquad x_n=y_{n-1}+Dy_n,$$

$$y_{n+1}=z_n+Dz_{n+1}, \qquad x_n=z_{n-2}+2Dz_{n-1}+D^2z_n. \tag{3.6}$$

From (3.4c) we obtain the recursion formula

$$z_{n+1}=z_{n-2}+3Dz_{n-1}+3D^2z_n$$

or

$$z_{n+3}=z_n+3Dz_{n+1}+3D^2z_{n+2}, \tag{3.7}$$

$$e^{-n}=z_{n-2}+2Dz_{n-1}+D^2z_n+(z_{n-1}+Dz_n)w+z_nw^2. \tag{3.8}$$

With the recursion formula (3.7) we can calculate z_n explicitly by Euler's generating function, viz.

$$\sum_{n=0}^{\infty} z_nu^n=z_0+z_1u+z_2u^2+\sum_{n=3}^{\infty} z_nu^n. \tag{3.9}$$

Substituting in (3.9) the first three initial values of z_n from the table (3.3) we obtain

$$\sum_{n=0}^{\infty} z_nu^n=u+3D^2u^2+\sum_{n=0}^{\infty} z_{n+3}u^{n+3}$$

$$=u+3D^2u^2+u^3\sum_{n=0}^{\infty}(z_n+3Dz_{n+1}+3D^2z_{n+2})u^n$$

$$=u+3D^2u^2+u^3\sum_{n=0}^{\infty} z_nu^n+3Du^2\sum_{n=0}^{\infty} z_{n+1}u^{n+1}+3D^2u\sum_{n=0}^{\infty} z_{n+2}u^{n+2}$$

$$=u+3D^2u^2+u^3\sum_{n=0}^{\infty} z_nu^n+3Du^2\left[\left(\sum_{n=0}^{\infty} z_nu^n\right)-z_0u^0\right]$$
$$+3D^2u\left[\left(\sum_{n=0}^{\infty} z_nu^n\right)-z_0u^0-z_1u\right]$$

$$=u+3D^2u^2+u^3\sum_{n=0}^{\infty} z_nu^n+3Du^2\sum_{n=0}^{\infty} z_nu^n+3D^2u\sum_{n=0}^{\infty} z_nu^n-3D^2u^2$$

$$=u+u^3\sum_{n=0}^{\infty} z_nu^n+3Du^2\sum_{n=0}^{\infty} z_nu^n+3D^2u\sum_{n=0}^{\infty} z_nu^n.$$

Thus

$$[1-(u^3+3Du^2+3D^2u)]\sum_{n=0}^{\infty} z_n u^n = u,$$

$$\sum_{n=0}^{\infty} z_n u^n = \frac{u}{1-(u^3+3Du^2+3D^2u)},$$

$$z_0u^0+\sum_{n=0}^{\infty} z_{n+1}u^{n+1} = \frac{u}{1-(u^3+3Du^2+3D^2u)},$$

$$\sum_{n=0}^{\infty} z_{n+1}u^n = \frac{1}{1-(u^3+3Du^2+3D^2u)}. \tag{3.10}$$

Choosing $u<1/9D^2$ we have $|u^3+3Du^2+3D^2u|<1$ so that from (3.10) we obtain

$$\sum_{n=0}^{\infty} z_{n+1}u^n = \sum_{i=0}^{\infty} (u^3+3Du^2+3D^2u)^i. \tag{3.11}$$

Using the multinomial formula we obtain

$$(u^3+3Du^2+3D^2u)^i = \sum_{y_1+y_2+y_3=i} \binom{i}{y_1,\, y_2,\, y_3} u^{3y_1}(3Du^2)^{y_2}(3D^2u)^{y_3},$$

$$(u^3+3Du^2+3D^2u)^i = \sum_{y_1+y_2+y_3=i} \binom{i}{y_1,\, y_2,\, y_3} 3^{y_2+y_3}D^{y_2+2y_3}u^{3y_1+2y_2+y_3}.$$

To find the coefficient of u^n in the expansion on the right side of (3.11) we obtain

$$z_{n+1} = \sum_{\substack{y_1+y_2+y_3=i\\ 3y_1+2y_2+y_3=n}} \binom{i}{y_1,\, y_2,\, y_3} 3^{y_2+y_3}D^{y_2+2y_3}. \tag{3.12}$$

The values of i in (3.12) are determined by two diophantine equations

$$y_1+y_2+y_3=i, \qquad 3y_1+2y_2+y_3=n,$$
$$n\geqslant 0, \quad y_1, y_2, y_3\geqslant 0; \quad [\tfrac{1}{3}n]\leqslant i\leqslant n. \tag{3.13}$$

4. New combinatorial identities

We have

$$e^n e^{-n} = (r_n+s_nw+t_nw^2)(x_n+y_nw+z_nw^2) = 1. \tag{4.1}$$

Multiplying on the right-hand side of (4.1) and substituting $w^3=D^3+1=m$, $w^4=mw$, we obtain

$$1 = r_nx_n+r_ny_nw+r_nz_nw^2+ms_nz_n+s_nx_nw+s_ny_nw^2+mt_ny_n+mt_nz_nw+t_nx_nw^2. \tag{4.2}$$

From (4.2) we obtain, by comparison of coefficients of equal powers of w (and because w is a third degree algebraic irrational)

$$\begin{aligned} r_n x_n + m t_n y_n + m s_n z_n &= 1, \\ s_n x_n + r_n y_n + m t_n z_n &= 0, \\ t_n x_n + s_n y_n + r_n z_n &= 0. \end{aligned} \tag{4.3}$$

In (4.3) we consider x_n, y_n, z_n as indeterminates, and r_n, s_n, t_n as coefficiential factors and

$$\Delta = \begin{vmatrix} r_n & mt_n & ms_n \\ s_n & r_n & mt_n \\ t_n & s_n & r_n \end{vmatrix}. \tag{4.4}$$

The determinant (4.4) is the norm of e^n. We have

$$\begin{aligned} e^n &= r_n + s_n w + t_n w^2, \\ e^n w &= m t_n + r_n w + s_n w^2, \\ e^n w^2 &= m s_n + m t_n w + r_n w^2. \end{aligned} \tag{4.5}$$

From (4.5) we obtain

$$N(e^n) = \begin{vmatrix} r_n & s_n & t_n \\ mt_n & r_n & s_n \\ ms_n & mt_n & r_n \end{vmatrix} = \begin{vmatrix} r_n & mt_n & ms_n \\ s_n & r_n & mt_n \\ t_n & s_n & r_n \end{vmatrix} = \Delta. \tag{4.6}$$

We shall now calculate $N(e^n)$.

$$N(e^n) = [N(e)]^n, \tag{4.7}$$

$$N(e) = N(w-D) = N[-(D-w)] = (-1)^3 N(D-w) = -N(D-w). \tag{4.8}$$

$$\begin{aligned} &D^3 + 1 = w^3, \qquad D^3 - w^3 = -1 \\ &(D-w)(D-\rho w)(D-\rho^2 w) = -1, \quad \rho = e^{2\pi i/3}, \end{aligned} \tag{4.9}$$

We thus obtain $N(e) = -N(D-w) = -1(-1) = 1$.

$$N(e) = N(e^n) = 1 = \Delta. \tag{4.10}$$

From (4.3), by Cramer's rule and $\Delta = 1$ we have

$$z_n = s_n^2 - r_n t_n. \tag{4.11}$$

Substituting in (4.11) the values for s_n and r_n from (2.9) we obtain

$$z_n = (t_{n+1} + D t_n)^2 - (t_{n+2} + 2D t_{n+1} + D^2 t_n) t_n$$

or

$$z_n = t_{n+1}^2 - t_n t_{n+2}, \quad n = 0, 1, \dots. \tag{4.12}$$

Substituting in (4.12) the values of z_n from (3.12) and t_n from (2.19), we obtain the first of the wanted combinatorial identities.

We return to the system of equations (4.3) and rearrange it, considering r_n, s_n, t_n as indeterminates where the determinant of the equations this time equals

$$\Delta_1 = \begin{vmatrix} x_n & mz_n & my_n \\ y_n & x_n & mz_n \\ z_n & y_n & x_n \end{vmatrix}. \tag{4.13}$$

As before $\Delta_1 = N(e^{-n}) = (N(e))^{-1} = 1^{-1} = 1$. In a similar way we obtain

$$t_n = y_n^2 - x_n z_n. \tag{4.14}$$

Substituting in (4.14) the values of x_n, y_n from (3.8) we obtain

$$t_n = (z_{n-1} + Dz_n)^2 - z_n(z_{n-2} + 2Dz_{n-1} + D^2 z_n).$$

or

$$t_n = z_{n-1}^2 - z_{n-2} z_n. \tag{4.15}$$

(4.15) supplies a second combinatorial identity. Both have the same structure.

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