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TRIGONOMETRIC IDENTITIES

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ABSTRACT. In this paper the author obtains new trigonometric identities of the form $2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-2} (1-\cos \frac{2\pi k}{p})^{p-1-k} = p^{p-2}$ which are derived as a result of relations in a cyclotomic field $\mathcal{R}(\rho)$, where $\mathcal{R}$ is the field of rationals and ρ is a root of unity.

Those identities hold for every positive integer $p \geq 3$ and any proof avoiding cyclotomic fields could be very difficult, if not insoluble. Two formulas

$$\sum_{k=1}^{\frac{p-1}{2}} (-1)^k \binom{p}{2k} \tan^{p-1-2k} \phi = 0 \qquad \text{and}$$

$$-1 + \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \left(\sum_{i=0}^{\frac{p-1-2k}{2}} \binom{p}{2k+2i} \binom{k+i}{k} \right) \cos^{p-2k} \phi = 0$$

stated only by Gauss in a slightly different form without a proof, are obtained and used in this paper in order to give some numeric applications of our new trigonometric identities.

KEY WORDS AND PHRASES. Trigonometric identities, cyclotomic field.
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0. INTRODUCTION

The trigonometric identities which are being obtained in this paper are a result of relations in a cyclotomic field $\mathcal{R}(\rho)$, where $\mathcal{R}$ is the field of rationals and ρ is a root of unity. The reader will be familiar with the primitive p-th roots of unity which are the p-1 different roots of the irreducible polynomial

$$x^{p-1} + x^{p-2} + \cdots + x + 1 = 0, \quad p \text{ a prime} > 2 \tag{0.1}$$

which we shall denote by ρ. As is well known

$$\rho = \cos\phi + i \sin\phi, \ \phi = \frac{2\pi k}{p}, \quad k = 1,2,\ldots,k-1 \tag{0.2}$$

Since the p-1 entities $\rho, \rho^2, \ldots, \rho^{p-1}$ form all the different roots of (0.1) we shall choose

$$\rho = \cos\phi + i\sin\phi, \quad \phi = \frac{2\pi}{p}, \qquad (k=1). \tag{0.3}$$

The cyclotomic fields have been substantially investigated and the author will make use of some comments. Yet there are still many open problems in this domain. The full group of fundamental units has not yet been revealed and this remains one of the most interesting problems.

1. IDENTITIES

The following important formula has been proved by Pollard [3] and many other authors and gives the discriminant $D(\rho)$ of the field $\mathcal{R}(\rho)$ in the following explicit form

$$\left.\begin{aligned} D(\rho) &= \prod_{1\le i<j\le p-1} (\rho^i - \rho^j)^2, \ \rho = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}, \\ D(\rho) &= (-1)^{\frac{p-1}{2}} p^{p-2}, \ \frac{2\pi}{p} = \phi, \quad p \text{ prime} > 2, \ \rho = \sqrt[p]{1} \end{aligned}\right\} \tag{1.1}$$

We shall first investigate

$$T = \prod_{1\le i<j\le p-1} (\rho^i - \rho^j); \quad T^2 = D(\rho) \tag{1.2}$$

We obtain from (1.2)

$$\begin{aligned} T = {} & (\rho - \rho^2)(\rho - \rho^3)(\rho - \rho^4) \quad \cdots \ (\rho - \rho^{p-1}) \\ & \qquad (\rho^2 - \rho^3)(\rho^2 - \rho^4) \ \cdots \ (\rho^2 - \rho^{p-1}) \\ & \qquad\qquad (\rho^3 - \rho^4) \ \cdots \ (\rho^3 - \rho^{p-1}) \\ & \qquad\qquad\quad \vdots \\ & \qquad\qquad\qquad\qquad (\rho^{p-2} - \rho^{p-1}) \\ = {} & \rho^{p-2}\rho^{2(p-3)}\rho^{3(p-4)}\cdots\rho^{(p-2)}(1-\rho)^{p-2}(1-\rho^2)^{p-3}(1-\rho^3)^{p-4}\cdots(1-\rho^{p-2}) \\ = {} & \prod_{k=1}^{p-2}\rho^{(p-1-k)k}(1-\rho^k)^{p-k-1} = \prod_{k=1}^{p-2}\rho^{(p-1-k)k}\prod_{k=1}^{p-2}(1-\rho^k)^{p-k-1} \end{aligned}$$

We shall calculate

$$\prod_{k=1}^{p-2}\rho^{(p-1-k)k} = \rho^{\sum_{k=1}^{p-2}(p-k-1)k} \tag{1.3}$$

We have

$$\sum_{k=1}^{p-2}(p-k-1)k = \sum_{k=1}^{p-2} pk - \sum_{k=1}^{p-2} k^2 - \sum_{k=1}^{p-2} k$$

$$= \frac{p(p-1)(p-2)}{2} - \frac{(2p-3)(p-1)(p-2)}{2} - \frac{(p-1)(p-2)}{2}$$

$$= \frac{p(p-1)(p-2)}{2} - \left(\frac{(2p-3+3)(p-1)(p-2)}{6} \right)$$

$$= \frac{p(p-1)(p-2)}{2} - \frac{p(p-1)(p-2)}{3} = \frac{p(p-1)(p-2)}{6}$$

$$\prod_{k=1}^{p-2} \rho^{(p-1-k)k} = \rho^{\frac{p(p-1)(p-2)}{6}} \tag{1.4}$$

Now

$$\rho = \cos \frac{2\pi}{p} + i \sin \frac{2\pi}{p} ; \rho^p = (\cos \frac{2\pi}{p} + i \sin \frac{2\pi}{p})^p$$

$$= \cos 2\pi + i \sin 2\pi = 1$$

$$\rho^{\frac{p(p-1)(p-2)}{6}} = (\rho^p)^{\frac{(p-1)(p-2)}{6}}$$

We shall presume here $p > 3$ and shall return to this case later. (In the case $p = 2$, the primitive unity roots are ± 1.) We then have $3 \nmid p$, hence $(p-1)(p-2) \equiv 0(6)$ so that either $p-1 \equiv 0(6)$ or $p-2 \equiv 0(3)$ and $p-1 \not\equiv 0(3)$.

Of course, since p is odd, $p-1 \equiv 0(2)$. So in any case $6|(p-1)(p-2)$, and since $\rho^p = 1$,

$$\rho^{\frac{p(p-1)(p-2)}{6}} = (\rho^p)^{\frac{(p-1)(p-2)}{6}} = 1^{\frac{(p-1)(p-2)}{6}} = 1, \text{ and from (1.4)}$$

$$\prod_{k=1}^{p-2} \rho^{(p-1-k)k} = 1. \tag{1.5}$$

We thus remain with

$$T = \prod_{k=1}^{p-2} (1 - \rho^k)^{p-k-1} \tag{1.6}$$

From (1.6) we obtain, since

$$\rho^k = \cos \frac{2\pi k}{p} + i \sin \frac{2\pi k}{p}, \frac{2\pi}{p} = \phi, \frac{2\pi k}{p} = k\phi$$

and from (1.6)

$$T = \prod_{k=1}^{p-2} [1 - (\cos k\phi + i \sin k\phi)]^{p-1-k}$$

$$= \prod_{k=1}^{p-2} [(1 - \cos k\phi) - i \sin k\phi]^{p-1-k}$$

$$= \prod_{k=1}^{p-2} (2 \sin^2 \frac{k\phi}{2} - i\, 2 \sin \frac{k\phi}{2} \cos \frac{k\phi}{2})^{p-1-k}$$

$$= \prod_{k=1}^{p-2} (-2\,i \sin \tfrac{k\phi}{2})^{p-1-k} (\cos \tfrac{k\phi}{2} + i \sin \tfrac{k\phi}{2})^{p-1-k}$$

$$T = \prod_{k=1}^{p-2} (-1)^{p-1-k} \prod_{k=1}^{p-2} i^{p-1-k} \prod_{k=1}^{p-2} (2 \sin \tfrac{k\phi}{2})^{p-1-k} \prod_{k=1}^{p-2} (\cos \tfrac{k\phi}{2} + i \sin \tfrac{k\phi}{2})^{p-1-k} \qquad (1.7)$$

We further have

$$\prod_{k=1}^{p-2} (-1)^{p-k-1} = (-1)^{\frac{(p-2)(p-1)}{2}}$$, and since p is odd $(-1)^{p-2} = -1$,

hence

$$\prod_{k=1}^{p-2} (-1)^{p-k-1} = (-1)^{\frac{p-1}{2}} \qquad (1.8)$$

$$\prod_{k=1}^{p-2} i^{p-1-k} = i^{\frac{(p-1)(p-2)}{2}} . \qquad (1.9)$$

$$\prod_{k=1}^{p-2} (\cos \tfrac{k\phi}{2} + i \sin \tfrac{k\phi}{2})^{p-1-k} = \prod_{k=1}^{p-2} (e^{\frac{i\,k\phi}{2}})^{p-1-k}$$

$$= \prod_{k=1}^{p-2} e^{\frac{i\phi}{2}(p-1-k)k} = e^{\frac{i\pi}{p}\,\frac{p(p-1)(p-2)}{6}} = e^{\frac{i\pi\,(p-1)(p-2)}{6}},$$

$$\left.\begin{aligned} &\prod_{k=1}^{p-2} (\cos \tfrac{k\phi}{2} + i \sin \tfrac{k\phi}{2})^{p-1-k} = (-1)^{\frac{(p-1)(p-2)}{6}} , \\ &(p-1)(p-2) \equiv 0(6). \end{aligned}\right\} \qquad (1.10)$$

With (1.8), (1.9), (1.10), (1.7) takes the form

$$T = (-1)^{\frac{p-1}{2}}\, i^{\frac{(p-2)(p-1)}{2}} (-1)^{\frac{(p-1)(p-2)}{6}} \prod_{k=1}^{p-2} (2 \sin \tfrac{k\phi}{2})^{p-1-k} . \qquad (1.11)$$

But from (1.2) $D(\rho) = T^2$, so with $(p-1)(p-2) \equiv 0(6)$

$$D(\rho) = (-1)^{p-1} (i^2)^{\frac{(p-2)(p-1)}{2}} \left((-1)^2\right)^{\frac{(p-1)(p-2)}{6}} \prod_{k=1}^{p-2} 2^{p-1-k} (2 \sin^2 \tfrac{k\phi}{2})^{p-1-k}$$

$$D(\rho) = (-1)^{\frac{p-1}{2}} \prod_{k=1}^{p-2} 2^{p-1-k} \prod_{k=1}^{p-2} (2 \sin^2 \tfrac{k\phi}{2})^{p-1-k}$$

$$D(\rho) = (-1)^{\frac{p-1}{2}}\, 2^{\frac{(p-2)(p-1)}{2}} \prod_{k=1}^{p-2} (1 \cos k\phi)^{p-1-k} = (-1)^{\frac{p-1}{2}}\, p^{p-2}$$

from (1.1).
Hence we obtain our identity

$$2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-2} (1-\cos k\phi)^{p-1-k} = p^{p-2} \tag{1.12}$$

or

$$2^{\frac{(p-1)(p-2)}{2}} \prod_{k=1}^{p-2} (1-\cos \frac{2\pi k}{p})^{p-1-k} = p^{p-2}. \tag{1.13}$$

It is interesting that the identity (1.13) is valid also for $p = 3$.

We have $p = 3$, $k = 1$, $2^{\frac{(p-1)(p-2)}{2}} = 2^1 = 2$

$$2\prod_{k=1}^{1}(1-\cos\frac{2\pi}{3})^1 = 2(1 - \cos 120^\circ) = 2(1 + \tfrac{1}{2}) = 3 = 3^{3-2}.$$

But the proof is not valid here, since $3|p$. We find directly, with $\rho = \cos 120^\circ + i \sin 120^\circ$ from (1.1)

$D(\rho) = (\rho - \rho^2)^2 = \rho^2(1 - 2\rho + \rho^2) = (\rho^2 - 2\rho^3 + \rho^4)$,
since $\rho^3 = 1$, $(\rho - 1)(\rho^2 + \rho + 1) = 0$, and for $\rho \neq 1$,

$$\rho^2 + \rho + 1 = 0,\ \rho^2 + \rho = -1.$$

Thus $\rho^2 - 2\rho^3 + \rho^4 = \rho^2 - 2 + \rho = -2 - 1 = -3$;

$$D(\rho) = (-1)^{\frac{3-1}{2}}\, 3^{3-2} = -3.$$

Also $[(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) - (\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})]^2$

$$= (2i\sin\frac{2\pi}{3})^2 = -4\cdot\frac{3}{4} = -3.$$

Now consider the Vandermonde determinant:

$$V = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \rho & \rho^2 & \cdots & \rho^{p-1} \\ 1 & \rho^2 & \rho^4 & \cdots & \rho^{2(p-1)} \\ . & . & . & \cdots & . \\ 1 & \rho^{p-1} & \rho^{2(p-1)} & \cdots & \rho^{(p-1)^2} \end{vmatrix}$$

$$= \prod_{0\le i<j\le p-1} (\rho^i - \rho^j) = T\cdot\prod_{j=1}^{p-1}(1-\rho^j),$$

T as in (1.6).

Multiplying this determinant by itself we obtain

$$V^2 = \begin{vmatrix} p & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & p \\ . & . & . & \cdots & . \\ 0 & 0 & p & \cdots & 0 \\ 0 & p & 0 & \cdots & 0 \end{vmatrix} = (-1)^{\frac{p-1}{2}} p^p.$$

In view of

$$\sum_{j=0}^{p-1} \rho^{kj} \cdot \rho^{j} = \sum_{j=0}^{p-1} \rho^{(k+1)j} = \begin{cases} 0 \text{ if } p \nmid k+1 \\ p \text{ if } p \mid k+1 \end{cases}$$

Since $\prod_{j=1}^{p-1} (x-\rho^{j}) = 1+x+x^2+\cdots+x^{p-1}$, for $x = 1$ we obtain $\prod_{j=1}^{p-1} (1-\rho^{j}) = p$. Consequently $V = pT$. Hence $|T^2| = p^{-2}|V^2| = p^{p-2}$. From the definition of T we obtain $|T^2| = \prod_{k=1}^{p-2} (|1-\rho^{k}|^2)^{p-k-1}$. But $|1-\rho^k|^2 = 2(1-\cos\frac{2k\pi}{p})$, and then $|T^2| = \prod_{k=1}^{p-2} (2(1-\cos\frac{2k\pi}{p}))^{p-k-1}$. Hence (1.13) follows.

Here we did not use the assumption that p is a prime number. Therefore (1.13) holds for every positive integer $p \geq 3$. (1.13) is an interesting identity and any proof avoiding cyclotomic fields could be very difficult, if not insoluble.

2. NUMERIC APPLICATION

We shall give a few examples for the identity (1.13). For this purpose, we need a short review of known formulas.

Let again be

$$\sqrt[p]{1} = \cos\frac{2\pi k}{p} + i \sin\frac{2\pi k}{p}, \quad k = 0,1,\ldots,p-1$$

$$\sqrt[p]{1} = (\cos\frac{2\pi}{p} + i \sin\frac{2\pi}{p})^k, \quad k = 0,1,\ldots,p-1$$

Choosing again $k = 1$, $\frac{2\pi}{p} = \phi$, we have

$$(\cos\phi + i\sin\phi)^p - 1 = 0$$

$$\cos^p\phi\,(i\tan\phi + 1)^p - 1 = 0$$

$$(i\tan\phi + 1)^p = \frac{1}{\cos^p\phi} = 0, \tag{2.1}$$

and for the imaginary part of (2.1) we obtain, after cancelling by i;

$$\tan^p\phi = \binom{p}{2}\tan^{p-2}\phi + \binom{p}{4}\tan^{p-4}\phi + \cdots + (-1)^{\frac{p-1}{2}} p\tan\phi = 0. \tag{2.2}$$

Excluding $\tan\phi = 0$, we can divide (2.2) by $\tan\phi$ and obtain the equation for $\tan\phi$

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{p}{2k} \tan^{p-1-2k}\phi = 0. \tag{2.3}$$

This formula was also stated by Gauss [1]. We shall still show how the $\cos\phi$ can be obtained from (2.3) since

$$\tan^2\alpha = \frac{1-\cos^2\alpha}{\cos^2\alpha}.$$

We obtain from (2.3)

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{p}{2k} \frac{(1-\cos^2\phi)^{\frac{p-1-2k}{2}}}{(\cos^2\phi)^{\frac{p-1-2k}{2}}} = 0. \tag{2.4}$$

Setting

$$\cos^2\phi = x \tag{2.5}$$

we obtain from (2.4)

$$\sum_{k=0}^{p-1} (-1)^k \binom{p}{2k} \frac{(1-x)^{\frac{p-1-2k}{2}}}{x^{\frac{p-1-2k}{2}}} = 0 \tag{2.6}$$

and in expanded form

$$\frac{(1-x)^{\frac{p-1}{2}}}{x^{\frac{p-1}{2}}} - \binom{p}{2} \frac{(1-x)^{\frac{p-3}{2}}}{x^{\frac{p-3}{2}}} + \binom{p}{4} \frac{(1-x)^{\frac{p-5}{2}}}{x^{\frac{p-5}{2}}} - + \cdots + (-1)^{\frac{p-1}{2}} p = 0$$

or

$$\sum_{k=0}^{p-1} (-1)^k (1-x)^{\frac{p-1-2k}{2}} x^k \binom{p}{2k} = 0. \tag{2.7}$$

This equation is of degree $\frac{p-1}{2}$ for $\cos^2\phi$. But there is another formula for $\cos\phi$ which is obtained in the following way. We obtain for the real part of

$$(\cos\phi + i\sin\phi)^p - 1 = 0$$

$$-1 + \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{p}{2k} \cos^{p-2k}\phi \sin^{2k}\phi = 0$$

or

$$-1 + \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{p}{2k} \cos^{p-2k}\phi\,(1-\cos^2\phi)^k = 0 \tag{2.8}$$

which is easily transformed into

$$-1 + \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \left(\sum_{i=0}^{\frac{p-1-2k}{2}} \binom{p}{2k+2i} \binom{k+1}{k} \right) \cos^{p-2k}\phi = 0. \tag{2.9}$$

We obtain for the first element under the sigma sign with $k = 0$

$$\sum_{i=0}^{\frac{p-1}{2}} \binom{p}{2i} = 1 + \binom{p}{2} + \binom{p}{4} + \cdots + \binom{p}{p-1} = 2^{p-1}.$$

Equation (2.9) is of degree p; one of its roots is, from $\frac{2\pi k}{p} = \phi = 0$ for $k = 0$, $\cos 0 = 1$, so the polynomial (2.9) is divisible by $\cos\phi - 1$. The quotient polynomial is of degree p-1, its p-1 roots $\frac{2\pi}{p}, \frac{2.2\pi}{p}, \ldots,$

$\frac{(p-1)2\pi}{p}$ are pairwise equal, viz. $\cos\frac{2\pi k}{p} = \cos(2\pi - \frac{2\pi k}{p})$, $k = 1,\ldots,p-1$, hence this quotient polynomial is the perfect square of a polynomial of degree $\frac{p-1}{2}$, the roots of which are $\cos k\phi$, $k = 1,\ldots,\frac{p-1}{2}$.

Formula (2.9) is also stated by Gauss [1] in a slightly different form.

Another way to obtain $\cos\phi$ directly is obtained in the following method investigated also by Perron for symmetric polynomial. We have the irreducible equation (0.1)

$$x^{p-1} + x^{p-2} + \cdots + x + 1 = 0.$$

Dividing by $x^{\frac{p-1}{2}}$ we obtain

$$\left(x^{\frac{p-1}{2}} + \frac{1}{x^{\frac{p-1}{2}}}\right) + \left(x^{\frac{p-3}{2}} + \frac{1}{x^{\frac{p-3}{2}}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1 = 0. \tag{2.10}$$

We obtain easily, setting

$$\left.\begin{aligned} & x + \frac{1}{x} = y, \\ & x^2 + \frac{1}{x^2} = y^2 - 2, \quad x^3 + \frac{1}{x^3} = y^3 - 3y, \ \ldots \\ & x^{\frac{p-1}{2}} + \frac{1}{x^{\frac{p-1}{2}}} = y^{\frac{p-1}{2}} + \cdots . \end{aligned}\right\} \tag{2.11}$$

But $x + \frac{1}{x} = \cos\phi + i\sin\phi + \cos\phi - i\sin\phi = 2\cos\phi = y$, so that a root of (2.10) is $2\cos\phi$.

3. TWO EXAMPLES

We shall first investigate

<u>Case A</u> $p = 5$.

Equation (2.3) yields

$$\tan^4\phi - \binom{5}{2}\tan^2\phi + \binom{5}{4} = 0;$$

Setting $\tan^2\phi = x$, we obtain

$$x^2 - 10x + 5 = 0$$

$$x = \tan^2\phi = 5 + 2\sqrt{5}$$

$$\frac{1}{\cos^2\phi} = 1 + \tan^2\phi = 6 + 2\sqrt{5},$$

$$\cos^2\phi = \frac{1}{6 + 2\sqrt{5}} = \frac{6 - 2\sqrt{5}}{16},$$

$$\cos\phi = \frac{\sqrt{5} - 1}{4}. \tag{3.1}$$

Equation (2.9) yields

$$-1 + \sum_{k=0}^{2} (-1)^k \left(\sum_{i=0}^{2-k} \binom{5}{2k+2i}\binom{k+1}{k} \right) \cos^{5-2k} \phi = 0,$$

hence

$$-1 + [1 + \binom{5}{2} + \binom{5}{4}]\cos^5 \phi - [\binom{5}{2} + \binom{5}{4}1]\cos^3 \phi + \binom{5}{4}\cos \phi = 0.$$

$$2^4 \cos^5 \phi - 20 \cos^3 \phi + 5 \cos \phi - 1 = 0 \qquad (3.2)$$

dividing (3.2) by $\cos \phi - 1$, we obtain

$$\frac{16 \cos^5 \phi - 20 \cos^3 \phi + 5 \cos \phi - 1}{\cos \phi - 1} = 16 \cos^4 \phi + 16 \cos^3 \phi - 4 \cos^2 \phi - 4 \cos \phi + 1$$

$$= (4 \cos^2 \phi + 2 \cos \phi - 1)^2 = 0.$$

$$4 \cos^2 \phi + 2 \cos \phi - 1 = 0 \text{ and } \cos \phi = \frac{\sqrt{5} - 1}{4}.$$

Finally we shall use formula (2.10) and obtain

$$\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1 = 0, \; x + \frac{1}{x} = y = 2 \cos \phi. \qquad (3.3)$$

Hence

$$y^2 - 2 + y + 1 = 0, \quad y^2 - y - 1 = 0, \quad y = \frac{\sqrt{5} - 1}{2}$$

$$\cos \phi = \frac{y}{2} = \frac{\sqrt{5} - 1}{4}, \text{ as in (3.1).}$$

We shall now verify identity (1.13) for $p = 5$ and prove, with $\frac{2\pi}{5} = \phi$

$$2^6(1 - \cos \phi)^3(1 - \cos 2\phi)^2(1 - \cos 3\phi) = 5^3, \qquad (3.4)$$

$$\cos \phi = \frac{\sqrt{5} - 1}{4}.$$

We have

$$1 - \cos \phi = \frac{5 - \sqrt{5}}{4} = \frac{\sqrt{5}\,(\sqrt{5} - 1)}{4}$$

$$(\sqrt{5} - 1)^3 = (6 - 2\sqrt{5})(\sqrt{5} - 1) = 8\sqrt{5} - 16 = 8(\sqrt{5} - 2)$$

$$(1 - \cos \phi)^3 = \frac{5\sqrt{5} \cdot 8(\sqrt{5} - 2)}{64} = \frac{5\sqrt{5}\,(\sqrt{5} - 2)}{8}$$

$$(1 - \cos 2\phi)^2 = (2 \sin^2 \phi)^2 = 4\left(1 - \frac{6 - 2\sqrt{5}}{16}\right)^2 = \frac{4(10 + 2\sqrt{5})^2}{256} =$$

$$4 \cdot 4 \cdot \frac{5(\sqrt{5} + 1)^2}{256} = \frac{5(\sqrt{5} + 1)^2}{16} = \frac{5(3 + \sqrt{5}}{8}$$

$$1 - \cos 3\phi = 1 - 4\cos^3 \phi + 3 \cos \phi = 1 - \frac{\sqrt{5} - 2}{2} + \frac{3\sqrt{5} - 3}{4} = \frac{5 + \sqrt{5}}{4}.$$

Hence $2^6(1 - \cos \phi)^3(1 - \cos 2\phi)^2(1 - \cos 3\phi) =$

$$2^6 \cdot \frac{5\sqrt{5}(\sqrt{5} - 2)}{8} \cdot \frac{5(\sqrt{5} + 3)}{8} \cdot \frac{\sqrt{5}\,(\sqrt{5} + 1)}{4} = \frac{2^6 \cdot 5^3 \cdot 4}{8 \cdot 8 \cdot 4} = 5^3,$$

which proves (3.4).

<u>Case B</u> $p = 7$.

Here it will be extremely difficult to state explicitly $\cos \frac{2\pi}{7} = \cos \phi$, and the more so to verify our identity (1.13). For we obtain from (2.3)

$$\tan^6 \phi - \binom{p}{2}\tan^4 \phi + \binom{p}{4}\tan^2 \phi + \binom{p}{6} = 0,$$

$\tan^6\phi - 21\tan^4\phi + 35\tan^2\phi - 7 = 0$, and

with $\tan^2\phi = y$,

$$y^3 - 21y^2 + 35y - 7 = 0. \qquad (3.5)$$

From (2.9) we obtain

$$-1 + 2^6\cos^7\phi - [\tbinom{7}{2} + \tbinom{7}{4}2 + \tbinom{7}{6}3]\cos^5\phi + [\tbinom{7}{4} + \tbinom{7}{6}\tbinom{3}{2}]\cos^3\phi - \tbinom{7}{6}\cos\phi = 0. \qquad (3.6)$$

$$64\cos^7\phi - 112\cos^5\phi + 56\cos^3\phi - 7\cos\phi - 1 = 0. \qquad (3.7)$$

Dividing (3.7) by $\cos\phi - 1$ yields

$$\frac{64\cos^7\phi - 112\cos^5\phi + 56\cos^3\phi - 7\cos\phi - 1}{\cos\phi - 1} = 64\cos^6\phi + 64\cos^5\phi - 48\cos^4\phi - 48\cos^3\phi + 8\cos^2\phi + 8\cos\phi + 1 = (8\cos^3\phi + 4\cos^2\phi - 4\cos\phi - 1)^2$$

$= 0$. $8\cos^3\phi + 4\cos^2\phi - 4\cos\phi - 1 = 0$, setting $\cos\phi = \frac{y}{2}$

$$y^3 + y^2 - 2y - 1 = 0. \qquad (3.8)$$

We finally get from (2.10)

$$\left(x^3 + \frac{1}{x^3}\right) + \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1 = 0; \qquad (3.9)$$

$$2\cos\phi = x + \frac{1}{x} = y, \quad \cos\phi = \frac{y}{2};$$

(3.9) yields

$$y^3 - 3y + y^2 - 2 + y + 1 = 0,$$

$y^3 + y^2 - 2y - 1 = 0$, which is equation (3.8).

We leave it to the reader to calculate $\cos\phi$ from (3.5) or (3.8), and then to verify our identity (1.13).

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